UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

October 2010 3 hours

Instructions: Answer completely as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

(1) Find all ring homomorphisms $f\colon \mathbb{Z} \to \mathbb{Z}/18\mathbb{Z}$.

(2) For $n \geq 2$, characterize the $n \times n$ matrices over $\mathbb{C}$ which commute *only* with diagonalizable matrices.

(3) Show that any group of order 10 is either a cyclic group or a dihedral group.

(4) (a) Let G be a group. Show that the conjugation homomorphism $c\colon G \to \mathrm{Aut}(G)$ is injective if and only if the centre of G is trivial.
(b) If G is simple and nonabelian, is c necessarily an isomorphism? Prove or give a counterexample.

(5) Suppose that A and B are 4×4 matrices over $\mathbb{C}$ with the same minimal polynomial, characteristic polynomial, and at least two distinct eigenvalues. Prove that A and B are similar. Find an example of two 5×5 matrices over $\mathbb{C}$ with the same properties that are not similar.

(6) Let R be an integral domain. For an R-module M, let $M^* = \mathrm{Hom}_R(M, R)$.
(a) Verify that the function $i_M\colon M \to M^{**}$ given by
$$i_M(m)(f) = f(m)$$
for $m \in M$ and $f \in M^*$ is a R-module homomorphism, for any M.
(b) Show that i_M is injective if and only if M is torsion-free. (Assume M is finitely generated here.)
(c) If R is a PID, show that i_M is an isomorphism if M is torsion-free.
(d) Give an example of a ring R and R-module M for which $i_M = 0$.
(e) Give an example of a ring R and R-module M for which i_M is injective but not surjective.

(7) Show that, for positive integers m, n,
$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$$
as abelian groups, where $d = \gcd(m, n)$.

(8) Let G be a finite group of order $504 = 2^3 \cdot 3^2 \cdot 7$
(a) Show that G cannot be isomorphic to a subgroup of the alternating group A_7.
(b) If G is simple, determine the number of Sylow 3-subgroups.

(9) Let E be a splitting field of $x^3 - 2$ over the rationals $\mathbb{Q}$ and assume that E is a subfield of $\mathbb{C}$. Let $F = E \cap \mathbb{R}$ be the real subfield and note that $F = \mathbb{Q}[\sqrt[3]{2}]$.
(a) Show that $\mathrm{Gal}(E/\mathbb{Q})$ contains an element σ with the property that all elements of F fixed by σ are rational.
(b) Let $a \in F$ and suppose $a^3 \in \mathbb{Q}$. Show that one of $a, a\sqrt[3]{2}$ or $a\sqrt[3]{4}$ is contained in $\mathbb{Q}$.
(c) Prove that $\sqrt[3]{3} \notin E$.