Algebra Qualifying Exam
UWO, September 2011

Time: 3 hours. No aids are allowed. Maximum mark is 100. Justify all your steps and carefully write them down.

1. (*10 marks*) Let U be a unitary matrix ($U^*U = UU^* = I$). Show that

$$\lim_{n\to\infty} \frac{I + U + U^2 + \cdots + U^n}{n} = P,$$

where P is the orthogonal projection onto the subspace $\mathrm{Ker}\,(I - U)$. (Hint: use the fact that U is diagonalizabale in an orthonormal basis and consider various eigenspaces of U.)

2. (*10 marks*) Let $\mathrm{GL}_n(\mathbb{F}_q)$ denote the group of invertible n by n matrices over a finite field with q elements. Find the numbers of elements of this group.

3. (*10 marks*) The *center* of an algebra A is the set of all $a \in A$ such that $ab = ba$ for all $b \in A$. Determine the center of $M_n(F)$, the algebra of n by n matrices over a field F.

4. (*20 marks*) The *exponential map* $\exp : M_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$, from complex matrices to invertible ones, is defined by

$$\exp(A) = \sum_{p=0}^{\infty} \frac{A^p}{p!}$$

a. Show that for any invertible matrix g and any matrix A, we have

$$\exp(gAg^{-1}) = g\exp(A)g^{-1}$$

b. Use a) to show that any invertible matrix with *distinct* eigenvalues is the exponential of a matrix.

c. Show that if u is a nilpotent matrix then $I - u$ is in the range of the exponential map and use this to show that the exponential map is surjective. (Hint: for the very last part you can use the Jordan canonical form theorem).

5. (*10 marks*) The *n-th cyclotomic polynomial* is defined as

$$\varphi_n(x) := \prod(x - \zeta_n)$$

where the product is taken over the set of primitive n-th roots of unity. (Recall that ζ_n is a primitive nth root of 1 if $\zeta_n^n = 1$ but $\zeta_n^i \neq 1$ for all $i \in \mathbb{N}, i < n$.) Thus for example $\varphi_1(x) = x - 1, \varphi_2(x) = x + 1, \varphi_4(x) = x^2 + 1, \ldots$. Show that:

a. $x^n - 1 = \prod_{d|n} \varphi_d(x)$.
b. Deduce from a) that $\varphi_n(x) \in \mathbb{Z}[x]$.

6. (*10 marks*) Let $d|n, d \neq n$. Show that $\varphi_n(x)$ divides the polynomial $\frac{x^n-1}{x^d-1}$ in $\mathbb{Z}[x]$.
(Hint: Factor the polynomials $x^n - 1$, $x^d - 1$ in $\mathbb{C}[x]$.)

7. (*10 marks*) Show that a group of order 75 has a normal Sylow subgroup.

8. (*10 marks*) Let $\omega = \frac{-1}{2} + \frac{\sqrt{-3}}{2}$. Show that

a) $\omega^3 = 1$
b) *The field extension* $\mathbb{Q}(\omega, \sqrt[3]{5})$ *is a Galois extension of* $\mathbb{Q}$.
c) *Determine the Galois group* $\mathrm{Gal}(\mathbb{Q}(\omega, \sqrt[3]{5})/\mathbb{Q})$.

9. (*10 marks*) Let R be a commutative ring with unit. Let I and J be ideals in R such that $I + J = R$. (I and J are called *coprime ideals.*) Show that there is a natural *ring isomorphism*

$$\frac{R}{I \cap J} \to \frac{R}{I} \oplus \frac{R}{J}$$

(This is an abstract form of the *Chinese remainder theorem*).