The University of Western Ontario
Department of Mathematics
Ph.D. Comprehensive Examination (Analysis)

September 22, 2011 3 hours

Instructions: More credit will be given for a complete solution than for several partial solutions.

(1) Prove *Schwarz's theorem*: Let $f(z)$ be analytic for $|z| \leq R$, $f(0) = 0$, and $|f(z)| \leq M$. Then

$$|f(z)| \leq \frac{M|z|}{R}$$

(2) Prove that $\displaystyle\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot (2n)} 2\pi.$

(3) Prove that all the roots of $p(z) = z^7 - 5z^3 + 12$ lie between the circle $|z| = 1$ and $|z| = 2$.

(4) Show that the function $f : \{z \in \mathbb{C} \ : \ |z| > 1\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{2}(z + \frac{1}{z})$$

is injective and find its image.

(5) Let f be a holomorphic function on the open unit disc. Suppose there exists an open set R on the unit circle with the property that $\lim f(z) = 1$, as z approaches R (z is in the disc). Prove that f is identically 1.

(6) a) Consider the series $\sum_{n=1}^{\infty} \alpha_n \beta_n$, $\alpha_n, \beta_n \in \mathbb{R}$. Prove that if $\{\alpha_n\}$ is a non-increasing sequence, and the partial sums $B_N = \sum_{n=1}^{N} \beta_n$ are uniformly bounded in absolute value by some $L > 0$, i.e., $|B_N| \leq L$, $N = 1, 2, \dots$, then

$$S_N = \left|\sum_{n=1}^{N} \alpha_n \beta_n\right| \leq L \cdot (|\alpha_1| + 2|\alpha_N|).$$

(*Hint:* transform the formula for S_N so that b_n are replaced by B_n.)

b) Use (i) to prove the Dirichlet test for convergence of series: the sum

$$\sum_{n=1}^{\infty} a_n b_n$$

converges whenever the sequence $B_N = \sum_{n=1}^{N} b_n$ is bounded, and $\{a_n\}$ is a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$.

c) Use the Dirichlet test for convergence to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin n$$

converges. (*Hint:* use $2 \sin u \sin v = \cos(u - v) - \cos(u + v)$.)

(7) Let $f(x)$ be a differentiable function of one real variable defined for $x > 0$. Suppose that for all $x > 0$

$$|f(x)| \leq \frac{C}{x^k},$$

where $C > 0$, and $k \geq 0$. Further, assume that k is the best possible rate of growth for f, i.e., the inequality does not hold for any smaller values of k (and any $C > 0$). Finally, suppose that the same estimate holds for $|f'(x)|$ possibly with a different constant C, but the same k. Prove that

$$\lim_{x \to 0+} f(x)$$

exists.

(8) Let X be the metric space of continuous functions on the interval $[0, 1]$ with the metric defined as

$$d(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|,$$

and let Y be the metric space defined on the same collection of functions but with the metric

$$\rho(f, g) = \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

Show that X is a complete metric space, while Y is a metric space which is not complete.