Ph.D. Comprehensive Examination (Algebra)
Department of Mathematics
The University of Western Ontario
January 2021 3 hours

Instructions: There will be little or no partial credit, so you should aim to solve some problems completely and correctly rather than attempting every problem. You should attempt at least one question from each topic. Justify all of your answers. Start each on a new page.

Linear Algebra

1. Let A be an $n \times k$ matrix and let B be a $k \times n$ matrix, both over $\mathbb{C}$.
   (a) Show that the characteristic polynomials of AB and BA satisfy
   $$\lambda^k p_{AB}(\lambda) = \lambda^n p_{BA}(\lambda).$$
   In particular, if $n = k$, then AB and BA have the same characteristic polynomial.

   (Hint: consider the matrices
   $$P = \begin{bmatrix} \lambda I & A \\ B & I \end{bmatrix} \text{ and } Q = \begin{bmatrix} I & 0 \\ -B & \lambda I \end{bmatrix},$$
   where I and 0 denote identity and zero matrices of appropriate sizes, and consider the determinants of PQ and QP.)

   (b) When $n = k$, are AB and BA necessarily similar? If so, prove it. If not, give a counterexample.

2. Let U, V and W be finite-dimensional inner product spaces and let $S : U \to V$ and $T : V \to W$ be linear maps with adjoints S^* and T^*. Define the linear map $R : V \to V$ by $R = SS^* + T^*T$. If
   $$U \xrightarrow{S} V \xrightarrow{T} W$$
   is exact (that is, $\text{image}(S) = \ker(T)$), show that $R : V \to V$ is invertible.

Rings and modules
All rings are commutative with unity.

3. Let R be an integral domain. A nonzero non-unit element $s \in R$ is said to be *adept* if, for every element $a \in R$, there exist $q, r \in R$ with $a = qs + r$ with r either 0 or a unit of R.
   (a) If $s \in R$ is adept, prove that the principal ideal (s) generated by s is maximal in R.
   (b) Determine the adept elements of the polynomial ring $\mathbb{Q}[x]$.

4. Let R be a PID and let F be a free module with basis $e_1, \ldots, e_n$. Fix $x_1, \ldots, x_n \in R$, not all zero, and consider the submodule M spanned by $b := x_1e_1 + \cdots + x_ne_n$. Assume that $F = M \oplus N$ for some submodule N. Prove that the ideal $(x_1, \ldots, x_n)$ is equal to R.

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Group theory

5. Let G be a group and let D be the subgroup of G generated by $\{g^2 \mid g \in G\}$. Show that G/D is an abelian group.

6. Let G be a group of order 24. Show that G has either a normal subgroup of order 8 or a normal subgroup of order 4.

Field theory

7. (a) Show that the set $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ is a field.
   (b) Show that there is a field extension L of $\mathbb{Q}$ which is Galois over $\mathbb{Q}$ and such that $\mathbb{Q} \subset \mathbb{Q}(\sqrt{5}) \subset L$, with $\mathrm{Gal}(L/\mathbb{Q})$ a cyclic group of order 4.
   (c) Show that there is no Galois extension L of $\mathbb{Q}$ such that the following are both true:
      (i) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{3}) \subset L$, and
      (ii) $\mathrm{Gal}(L/\mathbb{Q})$ is a cyclic group of order 4.

8. Consider the field $\mathbb{F}_{2^6}$ with 64 elements.
   (a) Draw the lattice of all subfields of $\mathbb{F}_{2^6}$.
   (b) Determine the number of elements of $\mathbb{F}_{2^6}$ which do not lie in any proper subfield of $\mathbb{F}_{2^6}$.
   (c) Use (b) to determine the number of irreducible polynomials of degree six over $\mathbb{F}_2$.

   Be sure to explain all parts fully.