Department of Mathematics, Western University

PhD Comprehensive Examination Part I: Analysis

January 19, 2021 9:00 a.m.–12 noon

Instructions: *There are eight questions. Focus on providing complete solutions to the questions you attempt. More credit will be given for a complete solution than for several partial solutions.*

1. Let D_r denote the closed disc $|z| \le r$. Let $f(z)$ be an analytic function on D_1 whose image is contained in D_r for some $r < 1$. Prove that $f(z)$ has a unique fixed point.

2. We define a function $f : \mathbb{R} \to \mathbb{R}$ to be *left upper semi-continuous* (LUSC) if for all $a \in \mathbb{R}$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < a - x < \delta$ then $f(x) \le f(a) + \varepsilon$.

   Fix a function $f : \mathbb{R} \to \mathbb{R}$ and let $f_n : \mathbb{R} \to \mathbb{R}$ be LUSC functions for $n = 1, 2, \ldots$. Prove that if the sequence $f_1, f_2, \ldots$ converges uniformly to f on $\mathbb{R}$, then f is also LUSC.

3. If $P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, what is the value of $\int_C P(z)\, d\bar{z}$?

4. Suppose that $f(z)$ is analytic on a closed curve γ (i.e. f is analytic in an open region that contains γ). Show that

$$\int_\gamma \overline{f(z)} f'(z)\, dz$$

   is purely imaginary.

5. Find the volume of the set $\{(x, y, z) \in (0, 2)^3 : xyz < 1\}$.

6. Let X be a non-empty set and $B(X)$ be the set of all bounded functions $f : X \to \mathbb{R}$. Then

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

   defines a metric on $B(X)$. (There is no need to prove this.) Prove that if $(B(X), d)$ is a separable metric space then X is a finite set.

7. Let h_n, $n \in \mathbb{N}$, be a sequence of functions analytic in an open region Ω. Suppose that

$$H(z) = \sum_{n \ge 1} h_n(z)$$

   is uniformly convergent in Ω. Prove that $H(z)$ is analytic in Ω.

8. Let $\mathcal{G}$ be the collection of all functions $g : [-1, 1] \to \mathbb{R}$, such that $|g(x) - g(y)| \le 5|x - y|$ for all $x, y \in [-1, 1]$, and $g(-1) = g(1) = 1$. Evaluate, with proof,

$$\inf_{g \in \mathcal{G}} \int_{-1}^{1} g(t)^2\, dt.$$