THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 20, 2021 *3 hours*

Instructions: There will be little or no partial credit, so you should aim to solve some problems completely and correctly rather than attempting every problem. The exam is divided into the Real and Complex analytic parts (problems 1–4 and 5–8, respectively). You should attempt *at least* two problems from each group.

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1. Use Stone-Weierstrass theorem to prove that any continuous function on $[0, 1]$ can be uniformly approximated by piece-wise continuous functions.

2. Give an example of a complete countable metric space. Prove that any complete metric space without isolated points is uncountable.

3. (1) Use Green's theorem to prove the following identity
$$\iint_D f \Delta g \, dA = \oint_C f(\nabla g) \cdot \vec{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA.$$
Here D is a domain in $\mathbb{R}^2_{(x,y)}$ bounded by a smooth curve C, f, g are functions of class $C^1(\overline{D})$, $\vec{n}$ is an outward normal vector to C, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator, dA is the surface area element, and ds is the arclength element.
(2) Use part (1) to prove that if f is a harmonic function on a smoothly bounded domain $D \in \mathbb{R}^2$ (i.e., $\Delta f \equiv 0$), and $f(x, y) = 0$ on the boundary of the domain D, then
$$\iint_D |\nabla f|^2 dA = 0.$$

4. Let $f : [0, 1] \to \mathbb{R}^3$ satisfy the condition
$$|f(t_1) - f(t_2)| \leq |t_1 - t_2|^{1/2},$$
for $t_1, t_2 \in [0, 1]$. Prove that the range of f has no interior in $\mathbb{R}^3$.

5. Let γ denote the boundary of the square with vertices $1/2$, $i/2$, $-1/2$, $-i/2$, traversed counterclockwise. Evaluate the integral

$$\int_\gamma \frac{e^{2/z}}{z-1}\,dz\,.$$

6. Let f be a function, which is continuous on the closed unit disc and analytic on the open unit disc. Suppose that $|f(z)| = 1$ whenever $|z| = 1$. Prove that the function f can be extended to a meromorphic function with at most a finite number of poles in the whole plane.
[Hint: Show first that the function $1/\overline{f(1/\bar{z})}$ is analytic in the set $\{z \in \mathbb{C} : |z| > 1\}$ (except a finite number of poles) and coincides with f on the unit circle.]

7. Let $f(z) = \dfrac{1}{z^2+1} - \dfrac{2}{z}$. Find all possible Laurent expansions of f about $z_0 = -i$ and determine their regions of convergence.

8. Evaluate $\displaystyle\int_0^\infty \frac{dx}{16+x^4}$.