PhD Comprehensive Exam (Algebra)
Department of Mathematics
June 2021

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. You should attempt at least one question from each topic.
4. Justify all your answers.

Linear Algebra

1. Let A be a square matrix over $\mathbb{C}$ such that $\bar{A}^T = A$. Prove that the eigenvalues of A are real.
2. Consider the linear operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by multiplication by the matrix
$$\begin{bmatrix} 3 & -1 & 0 \\ 4 & -1 & 0 \\ -12 & 18 & 5 \end{bmatrix}.$$
Recall that a subspace $W \subseteq \mathbb{C}^3$ is T-invariant if $T(W) \subseteq W$. How many T-invariant subspaces of $\mathbb{C}^3$ are there? Remember you must prove that your answer is correct.

Groups

3. Let A be a finite abelian group of order 480. Consider the map
$$\phi : A \to A \qquad \phi(x) = 4x.$$
What possible values could $|\operatorname{coker}(\phi)|$ take?
4. Let G be a finite group with normal subgroup N. Prove or disprove, G is isomorphic to a semidirect product of N and G/N.

Rings and Modules

5. Let R be a PID and $R^n = M$ a free module of finite rank over R. We consider a bilinear form b on M. In other words we are given a function
$$b : M \times M \to R$$
satisfying the following rules
   1. $b(x, -) : M \to M$ is a homomorphism of R-modules for all $x \in M$.
   2. $b(-, r) : M \to M$ is a homomorphism of R-modules for all $y \in M$.

   Consider $M^\perp = \{x \in M \mid b(x, y) = 0 \text{ for all } y \in M\}$. Show that $M^\perp \oplus N \cong M$ for some submodule N of M.
6. Let I be the ideal in $\mathbb{Z}[X]$ generated by $X^3 + X + 1$ and 3. Let $R = \mathbb{Z}[X]/I$. Consider the abelian group $R^\times$, that is the group of units in R. Decompose $R^\times$ as direct product of cyclic groups. That is find n_i so that
$$R^\times \cong \mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \times \ldots \times \mathbb{Z}/n_k.$$

Fields

7. Let p be a prime and let K be a field of order p^{100}. Exactly how many subfields does K have? Remember to prove your answer.
8. Consider the polynomial $f = X^3 - 2tX + t \in \mathbb{C}(t)[X]$, where $\mathbb{C}(t)$ is the field of fractions of $\mathbb{C}[t]$. What is the Galois group of f over $\mathbb{C}(t)$?