PhD Comprehensive Exam (Algebra)
Department of Mathematics
5 October 2023, 9:00 am - 12:00 pm in MC 108

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. You should attempt at least one question from each topic.
4. Justify all your answers.

Linear Algebra

1. Let V be a finite-dimensional vector space over some field F. Define $\phi\colon \mathrm{End}(V) \to \mathrm{End}(V)$ by $\phi(f) = fg - gf$, where $g \in \mathrm{End}(V)$ is fixed. Show that if g is nilpotent, then so is ϕ.

2. Let f be an endomorphism of a finite-dimensional vector space V over some field F. Assume that its minimal polynomial is t^5 and that its kernel is 3-dimensional. What are the possible values of $n = \dim V$?

Groups

3. Let G be a non-Abelian group of order p^3, where p is a prime. Prove that $Z(G) = G'$, where $Z(G)$ denotes the centre of G and G' denotes its commutator subgroup.

4. Let $p < q$ be primes. Prove that there exists a non-Abelian group of order pq^2 if and only if p divides $q^2 - 1$.

Rings and Modules

5. Prove that polynomials $f(x), g(x) \in \mathbb{Z}[x]$ are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal they generate in $\mathbb{Z}[x]$ contains a nonzero integer.

6. Let A be a (possibly infinite) generating set of a module M over a ring R with 1. First, prove that A contains a maximal subset of R-linearly independent elements. Then deduce that, in the case when R is a field, A contains a (possibly infinite) basis for the free module M. Give an example to show that the assumption that R is a field cannot be omitted here.

Fields

7. Let E be the splitting field of $f(X) = X^4 + 3X^2 + 4 \in \mathbb{F}_5[X]$. Determine the Galois group of $E/\mathbb{F}_5$.

8. (a) Factor $X^{12} - 1 \in \mathbb{Q}[X]$ into irreducibles (explicitly!).

   (b) Assume that $n \geq 3$ is odd. Show that $\Phi_{2n}(X) = \Phi_n(-X)$, where $\Phi_n(X)$ is the n-th cyclotomic polynomial.