

THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 12, 2023 3 hours

Instructions: There will be little or no partial credit, so you should aim to solve at least 5 problems completely and correctly rather than attempting every problem. Skim the questions at the start, so you can focus on the ones you feel most confident about. You should attempt at least two complex analysis questions and at least two real analysis questions.

1. Prove that the subspace of all functions of the form $f(x) + g(y)$, where f, g are continuous real valued functions on $[0, 1]$, is closed in the space $C(K)$ of all continuous real valued functions on

$$K = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1; 0 \leq y \leq 1\},$$

(equipped with the sup metric).

2. Let U be an open set in \mathbb{R}^n , $n \in \mathbb{N}$, and let

$$f : U \rightarrow \mathbb{R}^n$$

be a mapping of class C^1 such that the Jacobian determinant $\det D_f(x) \neq 0$ for every $x \in U$. Show that $f(U)$ is open.

3. Let K be a compact set in \mathbb{R}^k . Let $f_n : K \rightarrow \mathbb{R}^m$, $n \in \mathbb{N}$, be an equicontinuous family of functions. Prove: if $(f_n(x))$ converges pointwise to $f(x)$ on K , then $f_n \rightarrow f$ uniformly on K .

4. Evaluate the surface integral

$$\int_S y^2 dA$$

where S is the part of the unit sphere $x^2 + y^2 + z^2 = 1$ in \mathbb{R}^3 that lies above the cone $z = \sqrt{x^2 + y^2}$.

5. Find the value of λ , for which

$$\int_{|z|=1} e^{\sin(\lambda/z)} dz = 2023.$$

Justify your answer.

6. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Justify your answer.

7. Let $f(z) = \frac{2}{(z+1)^2} - \frac{3}{z}$. Find all possible Laurent expansions of f about $z_0 = 1$ and determine their regions of convergence.

8. Prove that there exists no function f analytic in the unit disc $\{z : |z| < 1\}$ and such that $|f(z)| \rightarrow \infty$ as $|z|$ increases to 1.