THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 16, 2023   *3 hours*

Instructions: Answer as many questions as you can. Give complete justifications. Aim for complete solutions (there will be little or no partial credit). You should attempt at least two complex analysis questions and at least two real analysis questions.

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1. Recall that a map of the form $f(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, is called a linear-fractional transformation. Show that any linear-fractional transformation maps circles to circles. (We treat lines as circles of infinite radius.)

2. Let $f(z)$ be a holomorphic function in a domain $D \subset \mathbb{C}$, and $z \in D$. Prove that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta,$$

where $r > 0$ is sufficiently small.

3. Let $f(z)$ be an entire function (holomorphic on $\mathbb{C}$) and satisfies $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Prove that $f = \text{const}$.

4. Compute the integral

$$\int_{|z+i|=3/2} \frac{\sin(1/z)}{1+z^2}\, dz.$$

5. A point e in a convex set $C \subseteq \mathbb{R}^n$ is called *extreme point* if it cannot be represented as $e = (1-\lambda)x + \lambda y$ for some distinct $x, y \in C$ and $\lambda \in (0,1)$.

Let A be an $m \times n$ matrix with rank m and columns denoted by $A_1, \ldots, A_n \in \mathbb{R}^m$ and let $b \in \mathbb{R}^m$. Define the set $F := \{x \in \mathbb{R}^n \mid Ax = b,\ \ x_i \geq 0 \text{ for all } i = 1, \ldots, n\}$ and let $x^* \in F$.

Show that if x^* is an extreme point of F, then the set of columns $\{A_j | x_j^* \neq 0\}$ is linearly independent.

6. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone decreasing function with

$$\int_0^\infty f(x)\, dx < \infty.$$

Show that $\lim_{x\to\infty} x f(x) = 0$.

7. Compute the integral

$$\int_0^\infty \frac{\log(x)}{x^2 + a^2}\, dx,$$

where $a > 0$ is a constant.

8. Let $\{x(i,j) : i,j \in \mathbb{N}\}$ be a doubly indexed set in a complete metric space (X,ρ), such that

$$\rho(x(i,j), x(k,\ell)) \leq \min\left\{ \max\left\{\frac{1}{i}, \frac{1}{k}\right\}, \max\left\{\frac{1}{j}, \frac{1}{\ell}\right\}\right\}.$$

Prove that the iterated limits $\lim_{i\to\infty} \lim_{j\to\infty} x(i,j)$ and $\lim_{k\to\infty} \lim_{\ell\to\infty} x(k,\ell)$ exist and are equal.