THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

Monday June 10, 2019
1:30 - 4:30 pm

The exam consists of two parts. Part I contains mandatory problems and covers basic materials, while Part II covers slightly more advanced materials at graduate course level.

This is **Part II**, for which **60% is required to pass.** You may use a calculator, pen, and pencil. **NO other aids** are allowed. Your calculator **must NOT** be capable of wireless communications or capable of storing and displaying large text files.

Part II. Do **six (6)** out of the ten (10) questions below.

A. Numerical Methods:

1. Consider that we want to solve

$$f(x) = x^2 + 4x^2 - 10 = 0$$

for a root in $[1, 2]$.

(1-a) Rewrite $f(x)$ to minimize the number of arithmetic operations needed to evaluate it.

(1-b) Apply two steps of Newton's method with $x_0 = 1.5$ to find a root. Show intermediate results so that your method of calculation can be verified.

(1-c) Two possible fixed point iteration schemes that could also be used to solve for the same root are:

$$s = g_1(x) = \frac{1}{2}(10 - x^3)^{1/2}$$

and

$$x = g_2(x) = \sqrt{\frac{10}{4 + x}}.$$

Which of these do you expect to converge to the root found in (b) faster? Why? Assume you are using the same initial guess (you should **not** iterate the schemes to answer this question)

(1-d) Define what is meant by "order of convergence" for a root finding method. What is the order of convergence of the methods in parts (b) and (c)? What, if anything, changes if you have a multiple root?

2. Consider the initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y), \qquad t \geq a, \qquad y(a) = \alpha.$$

(2-a) Write down the backward (implicit) Euler method for this problem.

(2-b) For $f(t,y) = y^2$ the method in (a) can be solved analytically for y_{n+1}, the approximation to the solution at $t = a + (n+1)h$, Show that the solution is not unique, and that time steps h cannot be arbitrarily large.

(2-c) Repeat part (b) for $f(t,y) = y^2 - t$.

(2-d) The implicit trapezoidal method is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right].$$

Would this improve the situation for the $f(t,y)$ in part (b)? Justify your answer.

B. Partial Differential Equations:

3. Suppose that $u(t,a)$ satisfies

$$\begin{cases} u_t(t,a) + u_a(t,a) = -d_1(a)u(t,a) - d_2(a)u^2(t,a), & a > 0, \quad t > 0 \\ u(t,0) = b(t), \quad t > 0; \\ u(0,a) = g(a), \quad a > 0 \end{cases}$$

Let $\tau > 0$ and assume that

$$d_1(a) = \begin{cases} \delta_1, & a \le \tau, \\ \mu_1, & a > \tau, \end{cases} \qquad \text{and} \quad d_2(a) = \begin{cases} \delta_2, & a \le \tau, \\ 0, & a > \tau. \end{cases}$$

Find that $u(t,a)$ for $t > a$.

4. Let $x \in \mathcal{R}$ be the spatial variable and $t > 0$ be the time variable.

(4-a) Explain why a function of these two variables of the form $u(t,x) = \phi(x - ct)$ accounts for the phenomenon of a *traveling wave.*

(4-b) What is meant by saying that a traveling wave $u(t,x) = \phi(x - ct)$ is a traveling wave *front*?

(4-c) Can the PDE $u_t(t,x) = u_{xx}(t,x) - u_x(t,x)$ have a *non-constant* traveling wave *front* solution? Show your work (not simple say yes or no) !

C. Computer Algebra:

5. Consider two general polynomials

$$p(x) = \sum_{k=0}^{n} p_k x^k , \quad \text{and} \quad q(x) = \sum_{k=0}^{m} q_k x^k ,$$

with $m \le n$.

(5-a) Describe what is meant by the pseudo-division of $p(x)$ by $q(x)$. What is the pseudo-remainder?

(5-b) For the following two particular polynomials

$$p(x) = 3x^2 + 11x - 5 , \quad \text{and} \quad q(x) = 2x^2 + 7x + 1 ,$$

calculate the pseudo-quotient and pseudo-remainder of $p(x)$ divided by $q(x)$.

6. The greatest common divisor (GCD) of two integers M and N is the largest integer that divides both M and N.

(6-a) Describe the Euclidean algorithm for finding the GCD of two integers.

(6-b) Use the Euclidean algorithm to find the GCD of the integers 115 and 161.

(6-c) Describe how the Euclidean algorithm can show that two integers are relatively prime. Illustrate this using the integers 24 and 35.

(6-d) Show that the Euclidean algorithm has its worst case behaviour, i.e. it takes the most steps to terminate) for numbers taken from the Fibonacci sequence (recall that Fibonacci numbers f_n obey $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$). Illustrate this with the numbers f_7 and f_8.

(6-e) For $p(x)$ and $q(x)$ defined above in Question 6, use the Euclidean algorithm for polynomials to find a common factor or show that they are mutually prime.

D. Theoretical Physics:

7. Consider a non-rotating planet of constant density ρ and a radius R. A very narrow tube is bored from the North to the South Pole.

(7-a) Use the Gauss's law version of Newton's gravitational law, $\vec{\nabla} \cdot \vec{F} = -4Gm\rho$, to show that *inside the tube*, the gravitational force is proportional to the distance from the center of the planet.

(7-b) If there is no air resistance of frictional drag, show that an object dropped into the tube will oscillate between the poles with the same period as an object orbiting just above it surface

8. The following equation of state (let's call it a *principle*)

$$U = \left(\frac{3^5 \cdot 5}{2^{11} \pi^5 k^4} \right)^{1/3} hc \, \frac{S^{4/3}}{V^{1/3}}$$

describes photons in thermodynamic equilibrium.

(8-a) Find a more "popular" form of the equation of state relating U/V to T only. Then determine another equation relating S/V to T alone.

(8-b) Use the principle equation to find an expression for U/V in terms of the pressure p.

(8-c) Deduce the chemical potential from the principle equation.

(8-d) Why are there so many equations of state? Is this something unique to photons? Explain.

E. Computational Material Sciences:

9. Explain the principles of the so-called *Equation Free* coarse-graining method. Describe the different steps that are needed.

10. Padé scheme for finite differences uses the following form

$$\beta f'_{i-2} + \alpha f'_{i-1} + f'_i + \alpha f'_{i+1} + \beta f'_{i+2} = c\frac{f_{i+3} - f_{i-3}}{6h} + b\frac{f_{i+2} - f_{i-2}}{4h} + a\frac{f_{i+1} - f_{i-1}}{2h},$$

where the index i denotes the nodes and f'_i is the finite difference approximation to the first derivative at node i, and a, b, c, α and β are coefficients. Derive the second order constraint for the above.