THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

22 May 2023
Part I: 9:00 am - 12:00 pm

Instructions: The comprehensive exam consists of two parts. This is Part I. Part I consists of mandatory problems and covers basic material. In Part I, 80% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART I: Do ALL of the questions in the following four sections.

Linear Algebra

LA1. Consider the matrix $A = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$ and compute

(a) the eigenvalues of A

(b) a basis for each eigenspace of A, and

(c) a diagonal matrix D and invertible matrix X such that $A = XDX^{-1}$

LA2. Find a basis for the subspace of $\mathbb{R}^4$ of all vectors perpendicular to both

$$v = (1, 1, 0, 0) \quad \text{and} \quad w = (1, 2, 3, 4).$$

LA3. (True or False) The set of invertible $n \times n$ matrices forms a vector subspace of the vector space of $n \times n$ matrices.

Calculus

CA1. Find $\lim_{x \to 0} \dfrac{\int_0^x (e^{t^2} - 1 + t^2)^2 dt}{x^4 \sin x}$.

CA2. Suppose $\dfrac{\sin x}{x}$ is an anti-derivative of $f(x)$, evaluate $\displaystyle\int x^3 f'(x)\, dx$.

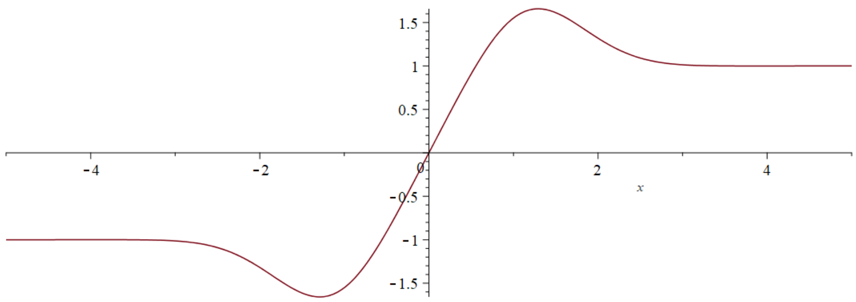
CA3. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{x-1}{x+1} \right)^{2n+1}$.

(a) Determine the interval of convergence.

(b) Find a formula for the sum function of this power series in the interval of convergence.

Numerical Methods

NA1. Consider the smooth function $f(x)$ whose graph is shown in the Figure where $f(x)$ has asymptotes $y = \pm 1$, is increasing on $-1.2 < x < 1.2$ and decreasing elsewhere.



(a) Approximately locate the one solution of $f(x) = 0$ using the graph. Illustrate the first 2 Newton steps starting with initial point $x = x_0 = 1$ by sketching on on a copy of the graph. Label x_1 and x_2 on your graph.

(b) Why do we expect Newton's iteration to converge for initial points sufficiently close to the approximate solution in (a)?

(c) Give some approximate intervals in x where Newton iteration will diverge when starting from an initial point in these intervals.

(d) Briefly, why is Newton's method for solving $f(x) = 0$ a consequence of Taylor's formula:

$$f(x + \Delta x) = f(x) + D(f)(x)\Delta x + O(2)$$

Briefly explain why this argument generalizes to systems of two equations in two variables by vectorizing (replacing x and $f(x)$ with vectors $\mathbf{x}$ and $\mathbf{f}$). What is $\mathbf{D}(\mathbf{f})(\mathbf{x})$ in this case?

NA2. Let $A = \begin{bmatrix} \epsilon & 3 \\ 2 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $0 < \epsilon \ll 1$ (i.e. ϵ is much smaller than 1).

(a) Show that reducing the system $A\mathbf{v} = \mathbf{b}$ by naive Gauss elimination (with no row changes) on the augmented matrix $[A|\mathbf{b}]$ yields

$$\left[\begin{array}{cc|c} \epsilon & 3 & 3 \\ 0 & (4 - \frac{6}{\epsilon}) & (2 - \frac{6}{\epsilon}) \end{array}\right]$$

(b) Show that working in any fixed precision by taking $\epsilon > 0$ sufficiently small the augmented matrix in (a) leads to the approximate solution $x_1 \approx 0$, $x_2 \approx 1$ where $\mathbf{v} = [x_1, x_2]^T$.

(c) Calculate the backward error of the approximate solution $x_1 \approx 0$, $x_2 \approx 1$. Note that the backward error is very large. However you can assume that condition number of A is approximately $8/3$. What does this imply about the method used in (a)-(b)?

(d) Use an alternative approach to (a)-(b) which in finite precision arithmetic yields an approximate solution of $A\mathbf{v} = \mathbf{b}$ with small backward error.

Ordinary Differential Equations

ODE1. Find the general solution to $t^3y' + 4t^2y = e^{-t}$ with $t > 0$.

ODE2. Find the general solution to $y^{(4)} - 4y'' - 5y = 7$.

ODE3. Consider $2y'' + xy' + 3y = 0$, and the point $x_0 = 0$. Find the recurrence relation for the coefficients of the power series solution.

ODE4. Let T be the temperature of a cup of coffee in a 70°F room. The coffee's temperature changes in proportion to the difference between its temperature and the room temperature. Write a differential equation for dT/dt, and solve the equation. Assume the constant of proportionality is k, and that the coffee is initially 200°F.