Robust trading strategies, pathwise Itô calculus, and generalized Takagi functions

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S&P 500 from 2006 through 2011

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Occam's razor suggests: Try working without a probability space and with minimal assumptions on price trajectories.

1. Continuous-time finance without probability

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Trading strategy (ξ, η):

- $\xi(t)$ shares of the risky asset
- $\eta(t)$ shares of a riskless asset

at time t.

Discounted portfolio value at time t:

$$V(t) = \xi(t)X(t) + \eta(t)$$

Key notion for continuous-time finance: self-financing strategy

If trading is only possible at times $0 = t_0 < t_1 < \cdots < t_N = T$, a strategy (ξ, η) is self-financing if and only if

$$(1) \qquad V(t_{k+1}) = V(0) + \sum_{i=0}^{k} \xi(t_i)\Big(X(t_{i+1}) - X(t_i)\Big), \qquad k = 0, \dots, N-1$$

How can we extend this definition to continuous trading?

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Now let $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions (i.e., $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \cdots$ and $\mathrm{mesh}(\mathbb{T}_n) \to 0$). Then (ξ, η) can be called self-financing (in continuous time) if we may pass to the limit in (1). That is,

$$V(t) = V(0) + \int_0^t \xi(s)\, dX(s), \qquad 0 \le t \le T,$$

where the integral should be understood as the limit of the corresponding Riemann sums:

$$\int_0^t \xi(s)\, dX(s) = \lim_{n\uparrow\infty} \sum_{t_i \in \mathbb{T}_n,\, t_i \le t} \xi(t_i)\Big(X(t_{i+1}) - X(t_i)\Big)$$

A special strategy

The following is a version of an argument from Föllmer (2001)

Proposition 1. *Let*

$$\xi(t) = 2\big(X(t) - X(0)\big) \qquad 0 \le t \le T.$$

Then $\int_0^t \xi(t)\, dX(t)$ *exists for all* t *as the limit of Riemann sums if and only if the* quadratic variation of X,

$$\langle X\rangle(t) := \lim_{N\uparrow\infty} \sum_{t_i\in\mathbb{T}_N,\, t_i\le t} \Big(X(t_{i+1}) - X(t_i)\Big)^2,$$

exists for all t. *In this case*

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exists for all t. *In this case*

$$\int_0^t \xi(s)\,dX(s) = \Big(X(t) - X(0)\Big)^2 - \langle X\rangle(t)$$

We always have $\langle X\rangle(t) = 0$ if X is of bounded variation or Hölder continuous for some exponent $\alpha > 1/2$ (e.g., fractional Brownian motion with $H > 1/2$)

Otherwise, the quadratic variation $\langle X\rangle$ depends on the choice of $(\mathbb{T}_n)$.

Additional arbitrage arguments showing the necessity of a well-behaved quadratic variation are due to Vovk (2012, 2015)

If $\langle X\rangle(t)$ exists and is continuous in t, Itô's formula holds in the following strictly pathwise sense (Föllmer 1981):

$$f(X(t)) - f(X(0)) = \int_0^t f'(X(s))\,dX(s) + \frac{1}{2}\int_0^t f''(X(s))\,d\langle X\rangle(s)$$

where

$$\int_0^t f'(X(s))\,dX(s) = \lim_{n\uparrow\infty} \sum_{t_i\in\mathbb{T}_N,\, t_i\le t} f'(X(t_i))\big(X(t_{i+1}) - X(t_i)\big)$$

is sometimes called the pathwise Itô integral or the Föllmer integral and $\int_0^t f''(X(s))\,d\langle X\rangle(s)$ is a standard Riemann–Stieltjes integral.

This formula was extended by Dupire (2009) and Cont & Fournié (2010) to a functional context

(Incomplete) list of financial applications of pathwise Itō calculus

- Strictly pathwise approach to Black–Scholes formula (Bick & Willinger 1994)
- Robustness of hedging strategies and pricing formulas for exotic options (A.S. & Stadje 2007, Cont & Riga 2016)
- Model-free replication of variance swaps (e.g., Davis, Obłój & Raval (2014))
- CPPI strategies (A.S. 2014)
- Functional and pathwise extension of the Fernholz–Karatzas stochastic portfolio theory (A.S., Speiser & Voloshchenko 2016)

For instance: hedging and pricing options

Bick & Willinger (1994) proposed a pathwise approach to hedging an option with payoff $H = h(X(T))$ for local volatility $\sigma(t, x)$

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Bick & Willinger (1994) proposed a pathwise approach to hedging an option with payoff $H = h(X(T))$ for local volatility $\sigma(t,x)$

For continuous h, solve the terminal-value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma(t,x)^2 x^2 \dfrac{\partial^2 v}{\partial x^2} = 0 \quad \text{in } [0,T) \times \mathbb{R}_+, \\ v(T,x) = h(x), \end{cases} \tag{2}$$

and let

$$\xi(t) := \frac{\partial}{\partial x} v(t, X(t))$$

Then the pathwise Itô formula yields that

$$v(0, X(0)) + \int_0^T \xi(t)\, dX(t) = h(X(T))$$

for any continuous trajectory X satisfying

$$\langle X \rangle(t) = \int_0^t \sigma(s, X(s))^2 X(s)^2\, ds \qquad \text{for all } t.$$

Extension to exotic options of the form

$$H = h(X(t_1), \ldots, X(t_n))$$

via solving an iteration scheme of the PDE (2), or for fully path-dependent payoffs

$$H = h((X(t))_{t \leq T})$$

via solving a PDE on path space (Peng & Wang 2016).

The preceding hedging argument leads to arbitrage-free pricing via establishing the absence of arbitrage in a strictly pathwise sense (Alvarez, Ferrando & Olivares 2013, A.S. & Voloshchenko 2016*b*)

2. In search of a class of test integrators

Let's fix the sequence of dyadic partitions of $[0,1]$,

$$\mathbb{T}_n := \{k2^{-n} \mid k = 0, \dots, 2^n\}, \qquad n = 1, 2, \dots$$

Goal: Find a rich class of functions $X \in C[0,1]$ that admit a prescribed quadratic variation along $(\mathbb{T}_n)$.

Of course one can take sample paths of Brownian motion or other continuous semimartingales—as long as these sample paths do not belong to a certain nullset A.

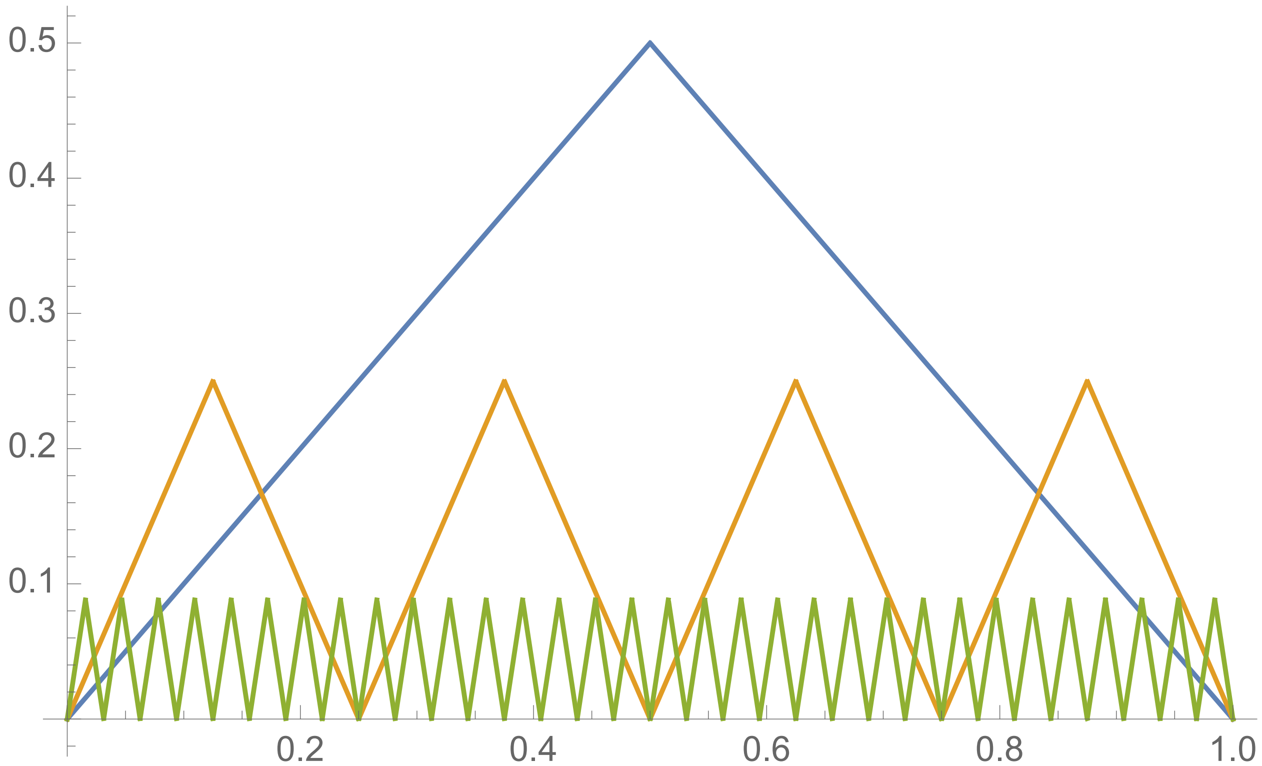
But A is not explicit, and so it is not possible to tell whether a specific realization X of Brownian motion does indeed admit the quadratic variation $\langle X\rangle(t) = t$ along $(\mathbb{T}_n)_{n\in\mathbb{N}}$.

Moreover, this selection principle for functions X lets a probabilistic model enter through the backdoor...

2.1 A result of N. Gantert

Recall that the Faber–Schauder functions are defined as

$$e_{0,0}(t) := (\min\{t, 1-t\})^+ \qquad e_{m,k}(t) := 2^{-m/2} e_{0,0}(2^m t - k)$$



Functions $e_{n,k}$ for $n = 0$, $n = 2$, and $n = 5$

Every function $X \in C[0,1]$ with $X(0) = X(1) = 0$ can be represented as

$$X = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m - 1} \theta_{m,k} e_{m,k}$$

where

$$\theta_{m,k} = 2^{m/2} \left(2X\left(\frac{2k+1}{2^{m+1}}\right) - X\left(\frac{k}{2^m}\right) - X\left(\frac{k+1}{2^m}\right) \right)$$

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Proposition 2. (Gantert 1994)

$$\langle X \rangle^n(t) := \sum_{t_i \in \mathbb{T}_n,\, t_i \le t} \big(X(t_{i+1}) - X(t_i)\big)^2$$

can be computed for $t = 1$ *as*

$$\langle X \rangle^n(1) = \frac{1}{2^n} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} \theta_{m,k}^2$$

2.2 Generalized Takagi functions with linear quadratic variation

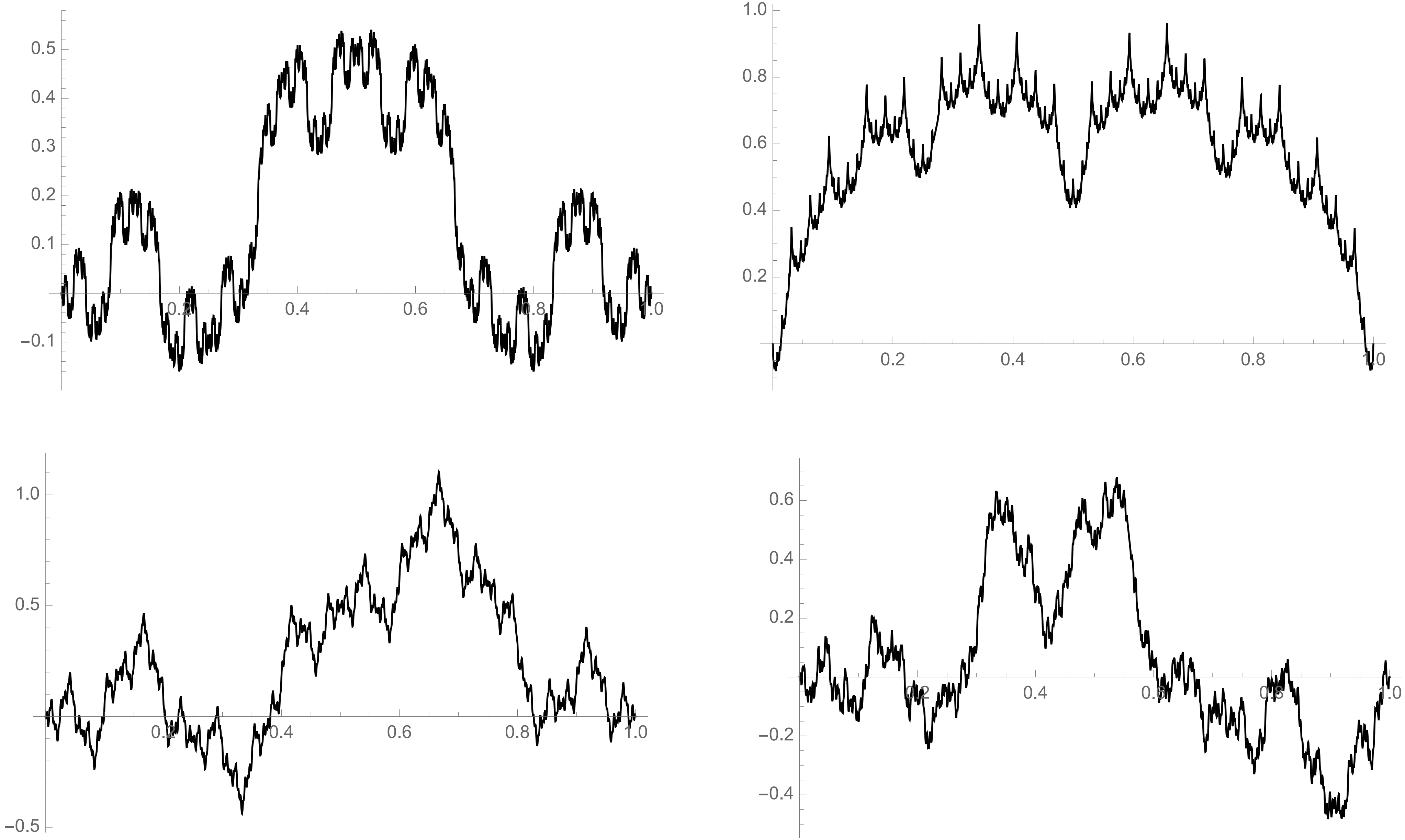
By letting

$$\mathscr{X} := \left\{ X \in C[0,1] \,\middle|\, X = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k} \text{ for coefficients } \theta_{m,k} \in \{-1,+1\} \right\}$$

(which is easily shown to be possible) we hence get a class of functions with $\langle X\rangle(1) = 1$ for all $X \in \mathscr{X}$.

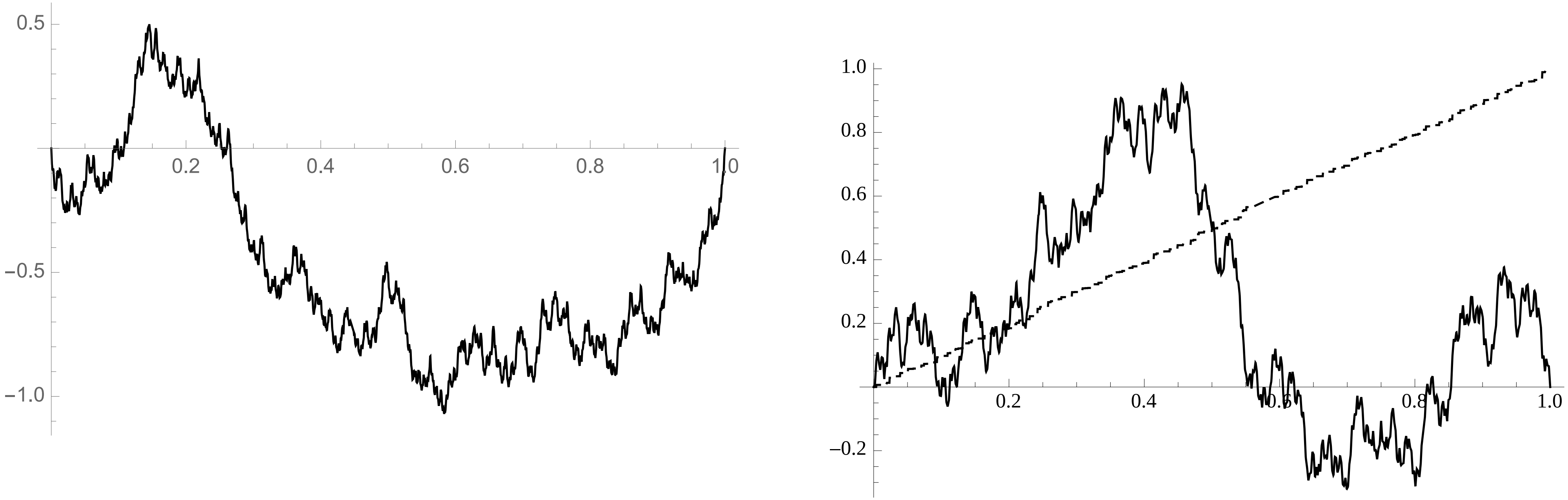
As a matter of fact:

Proposition 3. *Every $X \in \mathscr{X}$ has quadratic variation $\langle X\rangle(t) = t$ along $(\mathbb{T}_n)$.*



Functions in $\mathscr{X}$ for various (deterministic) choices of $\theta_{m,k} \in \{-1, 1\}$

Similarities with sample paths of a Brownian bridge



Plots of $X \in \mathscr{X}$ for a $\{-1,+1\}$-valued i.i.d. sequence $\theta_{m,k}$

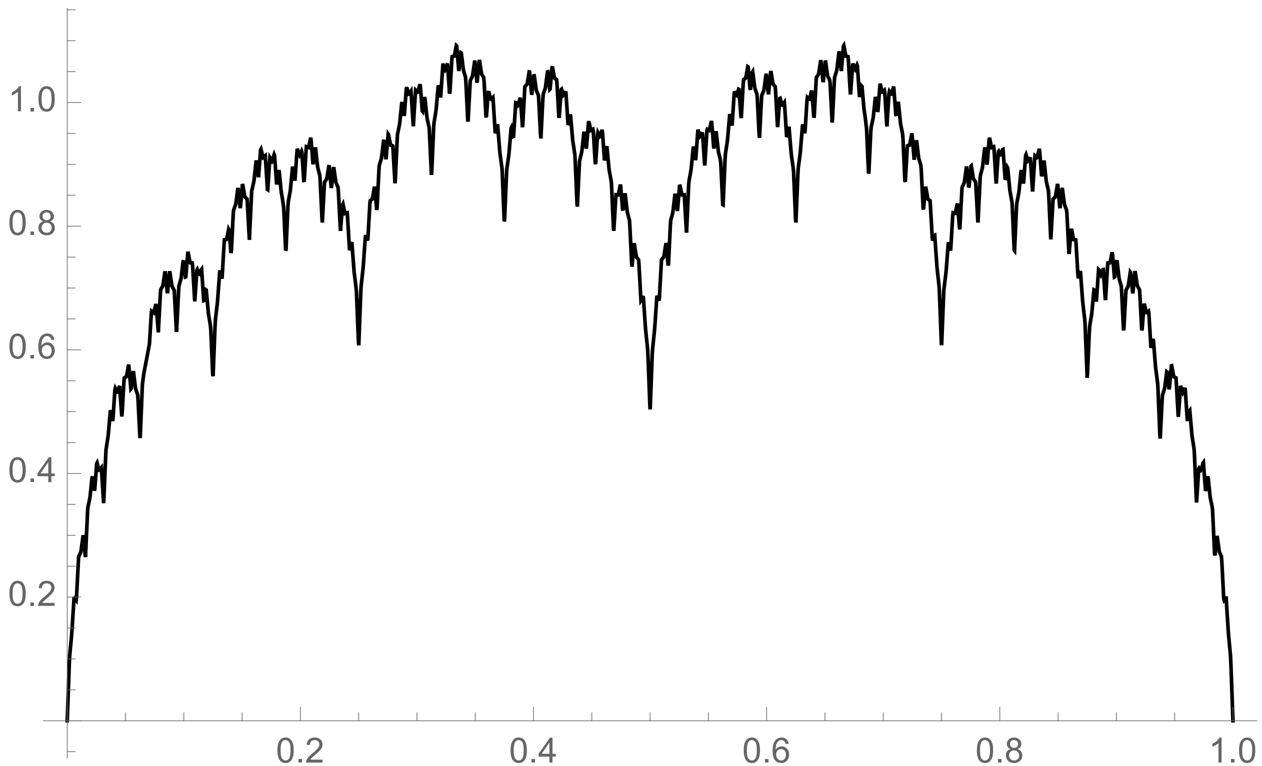
- Lévy–Ciesielski construction of the Brownian bridge
- Quadratic variation
- Nowhere differentiability (de Rham 1957, Billingsley 1982)
- Hausdorff dimension of the graph of $\widehat{X}$ is $\frac{3}{2}$ (Ledrappier 1992)

Link to the Takagi function and its generalizations

The specific function

$$\widehat{X} := \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} e_{m,k}$$

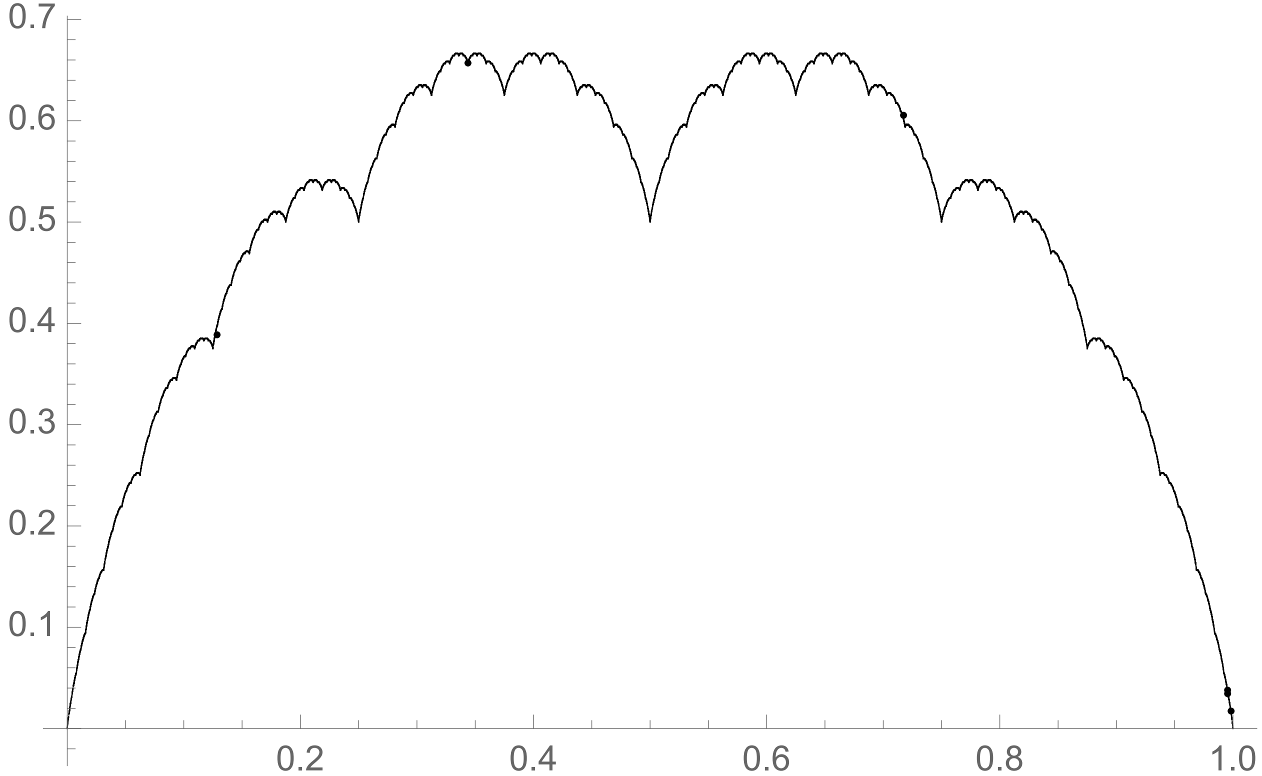
has some interesting properties.



The function $\widehat{X}$ is closely related to the celebrated Takagi function,

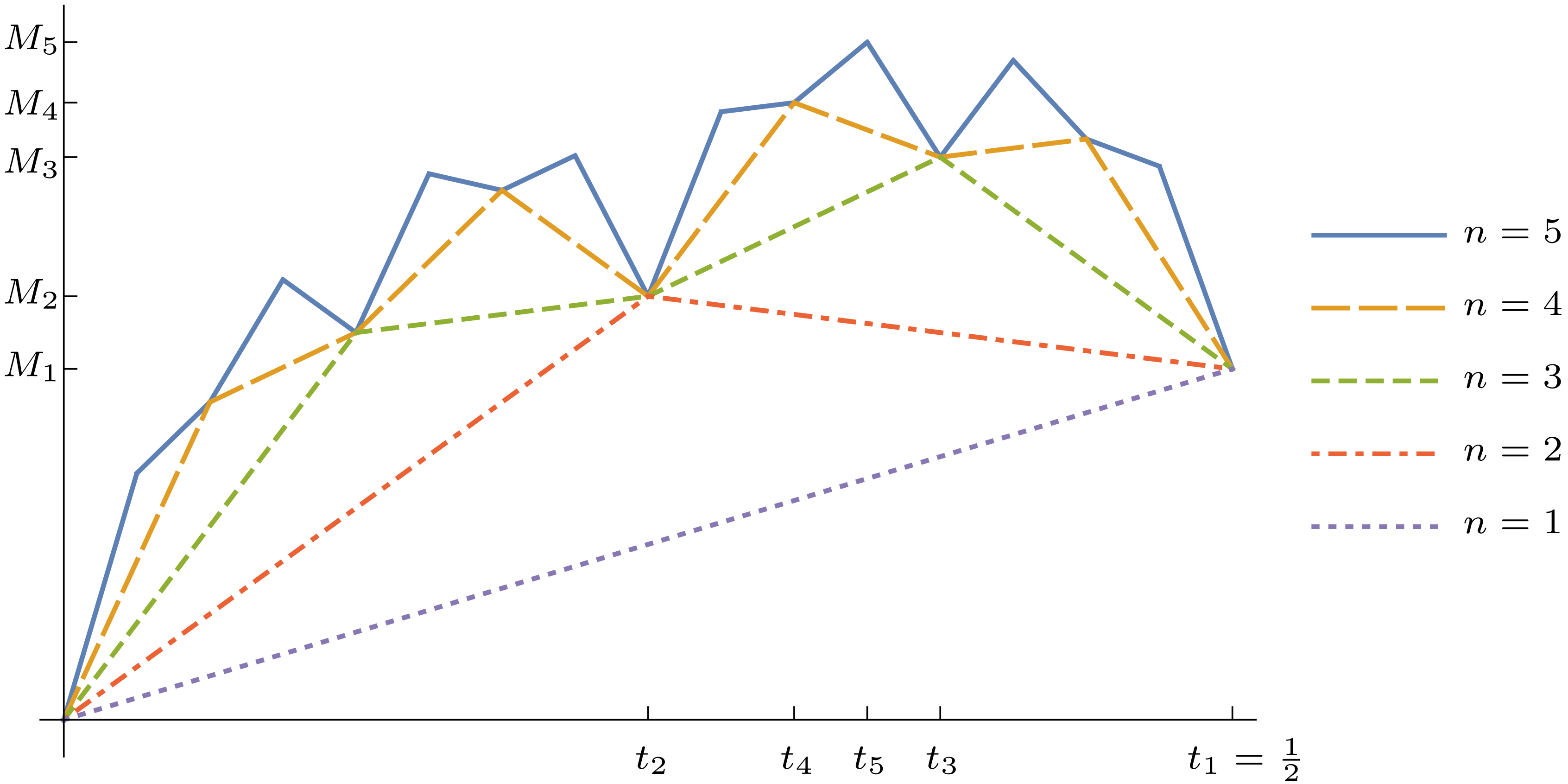
$$\widehat{X}^{(1)} = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} 2^{-m/2} e_{m,k}$$

which was first found by Takagi (1903) and independently rediscovered many times (e.g., by van der Waerden (1930), Hildebrandt (1933), Tambs–Lyche (1942), and de Rham (1957))



The maximum of $\widehat{\boldsymbol{X}}$

Kahane (1959) showed that the maximum of the Takagi function is $\frac{3}{2}$. For $\widehat{X}$, we need different arguments.



Functions $\widehat{X}^{n}(t) := \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} e_{m,k}(t)$ and their maxima on $[0, \frac{1}{2}]$

The preceding plot suggests the recursions

$$t_{n+1} = \frac{t_n + t_{n-1}}{2} \qquad \text{and} \qquad M_{n+1} = \frac{M_n + M_{n-1}}{2} + 2^{-\frac{n+2}{2}}$$

These are solved by

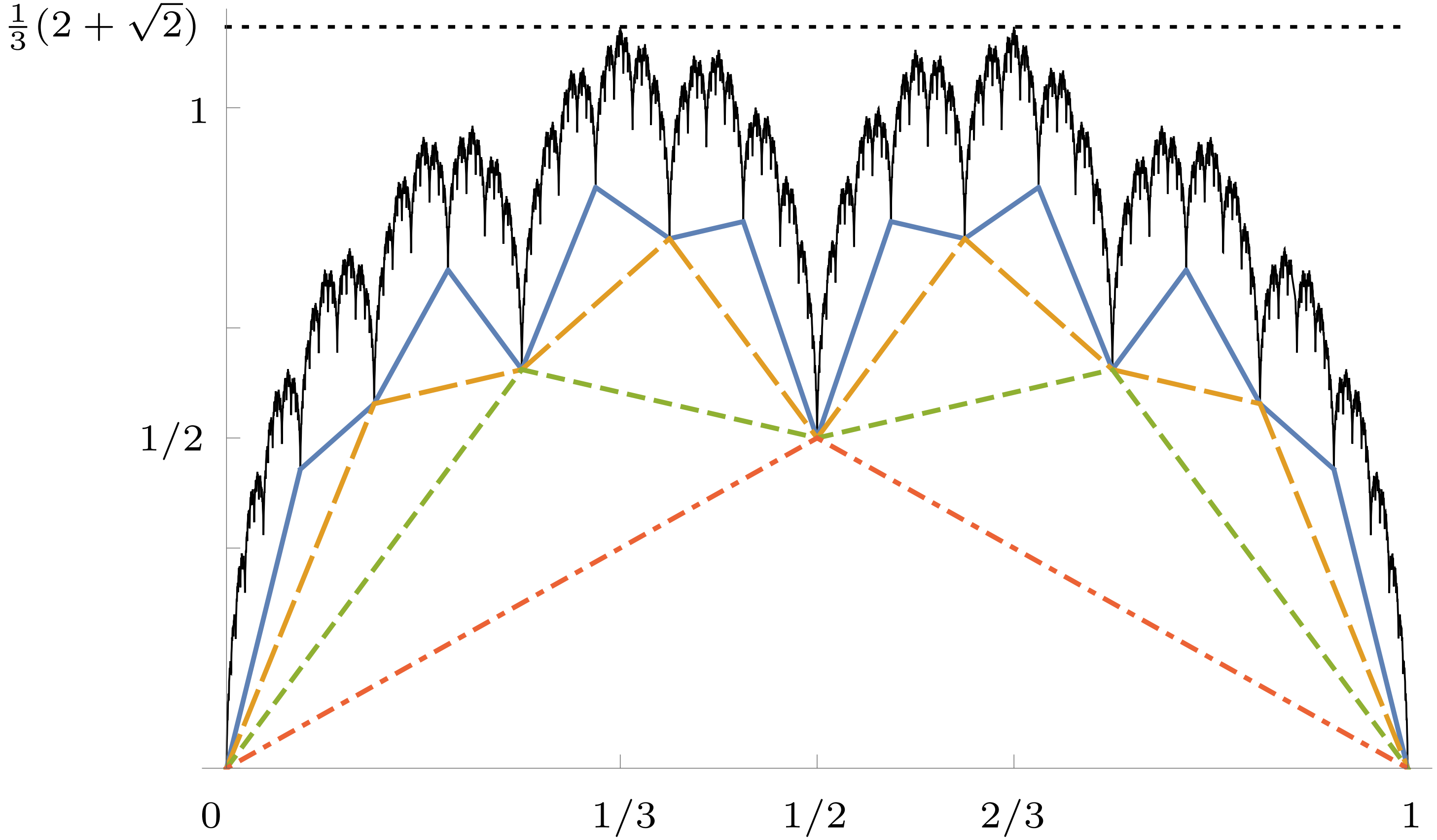
$$t_n = \frac{1}{3}(1 - (-1)^n 2^{-n}) \qquad \text{and} \qquad M_n = \frac{1}{3}\Big(2 + \sqrt{2} + (-1)^{n+1} 2^{-n}(\sqrt{2} - 1)\Big) - 2^{-n/2}$$

By sending $n \uparrow \infty$, we obtain:

Theorem 1. *The uniform maximum of functions in $\mathscr{X}$ is attained by $\widehat{X}$ and given by*

$$\max_{X\in\mathscr{X}} \max_{t\in[0,1]} |X(t)| = \max_{t\in[0,1]} \widehat{X}(t) = \frac{1}{3}(2+\sqrt{2}).$$

Maximal points are $t = \frac{1}{3}$ *and* $t = \frac{2}{3}$.

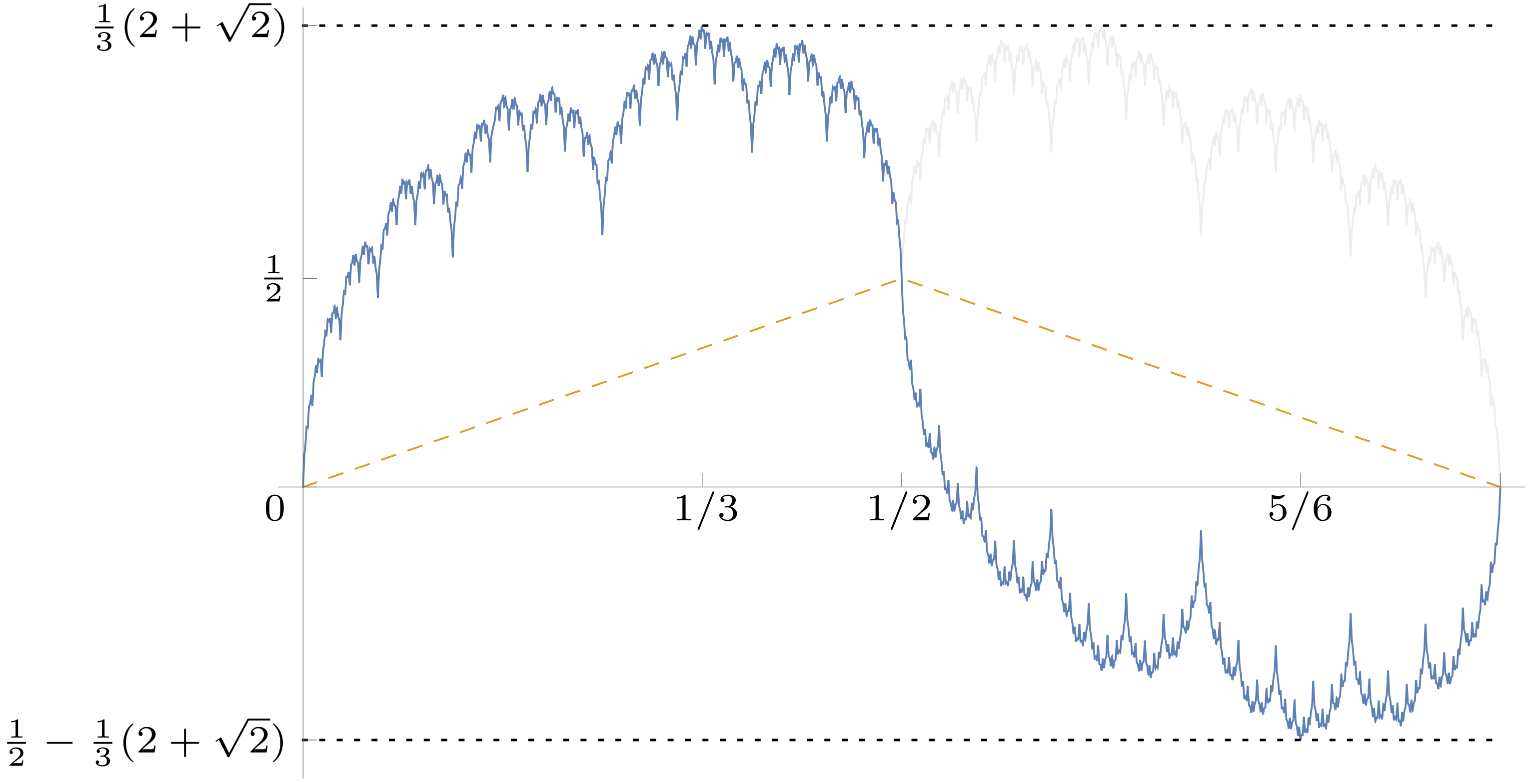


Corollary 1. *The maximal uniform oscillation of functions in $\mathscr{X}$ is*

$$\max_{X\in\mathscr{X}} \max_{s,t\in[0,1]} |X(t)-X(s)| = \frac{1}{6}(5+4\sqrt{2})$$

where the respective maxima are attained at $s = 1/3$, $t = 5/6$, and

$$X^* := e_{0,0} + \sum_{m=1}^{\infty}\left(\sum_{k=0}^{2^{m-1}-1} e_{m,k} - \sum_{\ell=2^{m-1}}^{2^m-1} e_{m,\ell}\right)$$



Uniform moduli of continuity

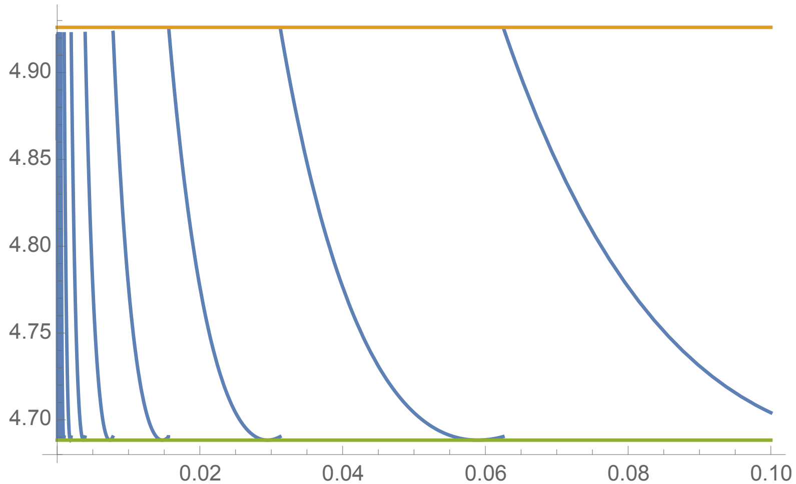
Kahane (1959), Kôno (1987), Hata & Yamaguti (1984), and Allaart (2009) studied moduli of continuity for (generalized) Takagi functions. However, their arguments are not applicable to the functions in $\mathscr{X}$.

Let

$$\omega(h) := \Big(1 + \frac{1}{\sqrt{2}}\Big) h 2^{\lfloor -\log_2 h \rfloor/2} + \frac{1}{3}(\sqrt{8} + 2) 2^{-\lfloor -\log_2 h \rfloor/2}$$

Then $\omega(h) = O(\sqrt{h})$ as $h \downarrow 0$. More precisely,

$$\liminf_{h\downarrow 0} \frac{\omega(h)}{\sqrt{h}} = 2\sqrt{\frac{4}{3} + \sqrt{2}} \qquad \limsup_{h\downarrow 0} \frac{\omega(h)}{\sqrt{h}} = \frac{1}{6}(11 + 7\sqrt{2})$$



Theorem 2 (Moduli of continuity).

(a) *The function* $\widehat{X}$ *has* ω *as its modulus of continuity. More precisely,*

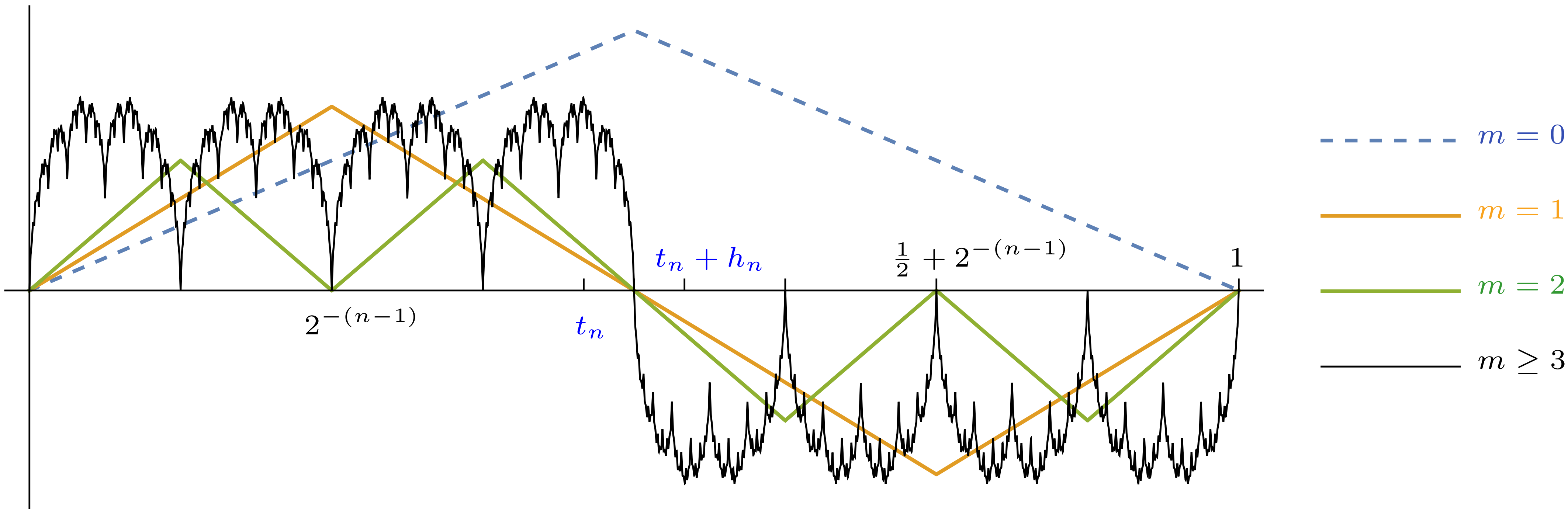
$$\limsup_{h\downarrow 0} \max_{0\le t\le 1-h} \frac{|\widehat{X}(t+h)-\widehat{X}(t)|}{\omega(h)} = 1$$

(b) *An exact uniform modulus of continuity for functions in* $\mathscr{X}$ *is given by* $\sqrt{2}\omega$*. That is,*

$$\limsup_{h\downarrow 0} \sup_{X\in\mathscr{X}} \max_{0\le t\le 1-h} \frac{|X(t+h)-X(t)|}{\omega(h)} = \sqrt{2}$$

Moreover, the above supremum over functions $X \in \mathscr{X}$ *is attained by the function* X^* *in the sense that*

$$\limsup_{h\downarrow 0} \max_{0\le t\le 1-h} \frac{|X^*(t+h)-X^*(t)|}{\omega(h)} = \sqrt{2}$$



The Faber–Schauder development of X^* is plotted individually for generations $m \leq n-1$ (with $n = 3$ here).

The aggregated development over all generations $m \geq n$ corresponds to a sequence of rescaled functions $\widehat{X}$.

$$\sqrt{2}\omega(h) = (\sqrt{2}+1)h2^{\lfloor -\log_2 h \rfloor /2} + \frac{2}{3}(2+\sqrt{2})2^{-\lfloor -\log_2 h \rfloor /2}$$

linear part self-similar part

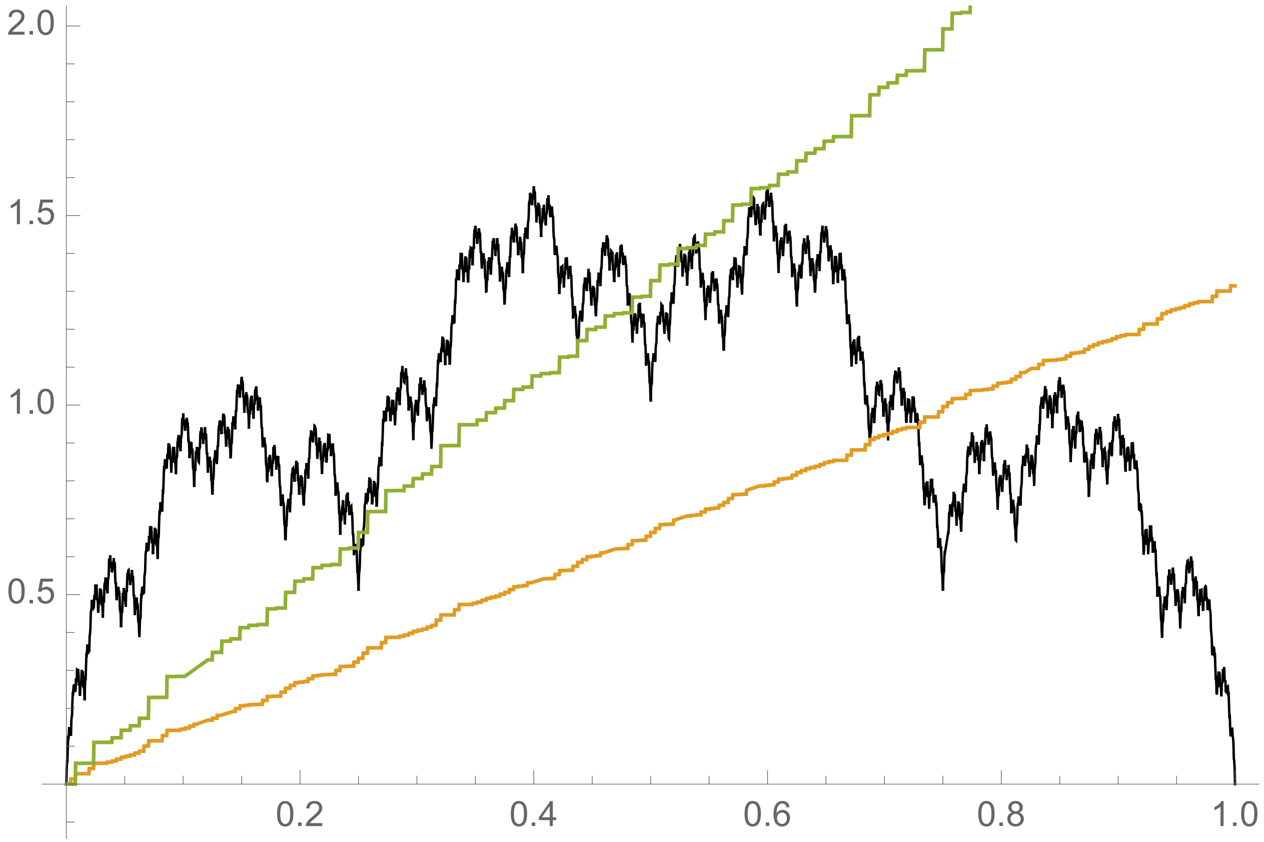
Consequences

- Functions in $\mathscr{X}$ are uniformly Hölder continuous with exponent $\frac{1}{2}$
- Functions in $\mathscr{X}$ have a finite 2-variation and hence can serve as integrators in rough path theory
- $\mathscr{X}$ is a compact subset of $C[0,1]$

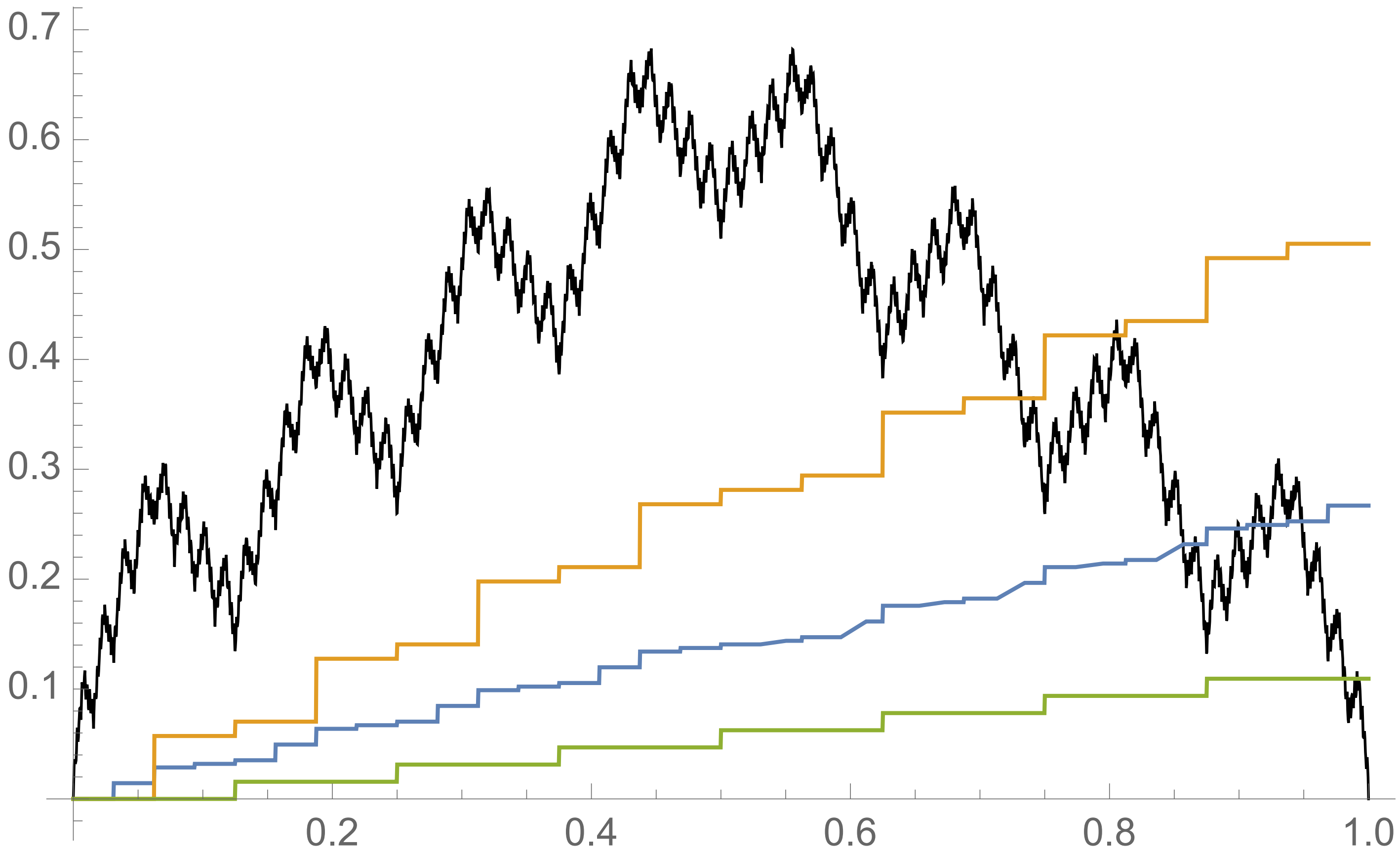
The linear hull of $\mathscr{X}$ is *not* a vector space

Proposition 4. *Consider the function $Y \in \mathscr{X}$ defined through $\theta_{m,k} = (-1)^m$. Then*

$$\lim_{n\uparrow\infty} \langle \widehat{X} + Y \rangle^{2n}(t) = \frac{4}{3}t \qquad \textit{and} \qquad \lim_{n\uparrow\infty} \langle \widehat{X} + Y \rangle^{2n+1}(t) = \frac{8}{3}t$$



The function $\widehat{X} + Y$ with $\langle \widehat{X} + Y \rangle^7$ and $\langle \widehat{X} + Y \rangle^8$



A function $Z \in \operatorname{span} \mathscr{X}$ with three distinct accumulation points for $\langle Z \rangle^n$

2.3 Vector spaces of functions with prescribed quadratic variation

The existence of a well-behaved covariation is needed, e.g., for describing multivariate price trajectories. We therefore need vector spaces of functions with prescribed quadratic variation. Here, we describe the constructions from Mishura & A.S. (2016)

Proposition 5. *Let $X \in C[0,1]$ have Faber–Schauder coefficients $\theta_{n,k}$. Then, for $t \in \bigcup_n \mathbb{T}_n$, the following conditions are equivalent.*

(a) *The quadratic variation $\langle X \rangle(t)$ exists*

(b) *The following limit exists,*

$$\ell(t) := \lim_{n\uparrow\infty} \frac{1}{2^n} \sum_{k=0}^{\lfloor (2^n-1)t \rfloor} \theta_{n,k}^2$$

In this case, we furthermore have

$$\langle X \rangle(t) = \ell(t)$$

Proof based on Proposition 2 and the Stolz–Cesàro theorem. ☐

Observe that

$$\frac{1}{2^n} \sum_{k=0}^{\lfloor (2^n - 1)t \rfloor} \theta_{n,k}^2$$

has the form of a Riemann sum for $\int_0^t f(s)^2 \, ds$ if we take

$$\theta_{n,k} := f(k 2^{-n})$$

Proposition 6. *If f is Riemann integrable on $[0,1]$, then*

$$X^f := \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} f(k2^{-n}) e_{m,k}$$

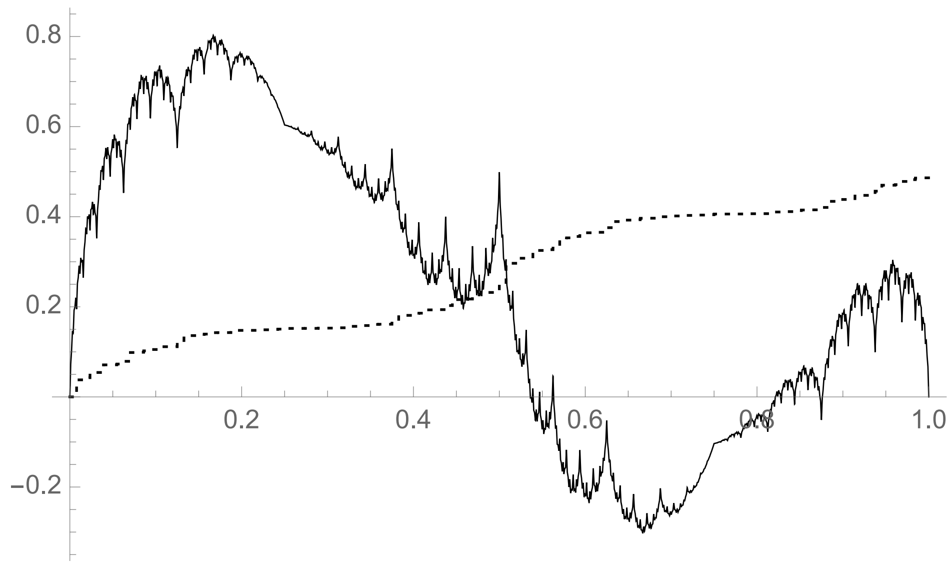
is a continuous function with quadratic variation

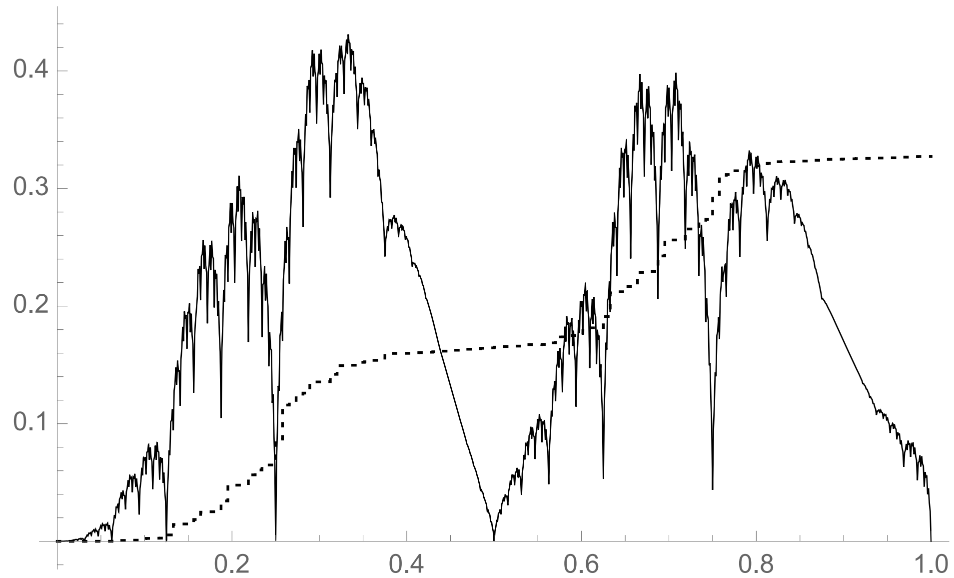
$$\langle X^f \rangle(t) = \int_0^t f(s)^2 \, ds$$

Thus, since the class $\mathscr{R}[0,1]$ of all Riemann integrable functions on $[0,1]$ is an algebra, the set

$$\{X^f \mid f \in \mathscr{R}[0,1]\}$$

is a vector space





Plots of the functions X^f for $f(t) := \cos 2\pi t$ (left) and $f(t) := (\sin 7t)^2$ (right). The dotted lines correspond to $\langle X^f \rangle^7$.

Proposition 7. *If f is Riemann integrable on $[0,1]$ and $\alpha > 0$ is irrational and fixed, then*

$$Y^{\alpha,f} := \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} f(\alpha k \bmod 1) e_{m,k}$$

is a continuous function with quadratic variation

$$\langle Y^{\alpha,f} \rangle(t) = t \int_0^1 f(s)^2 \, ds$$

Proof is based on Proposition 5 and Weyl's equidistribution theorem, which implies that

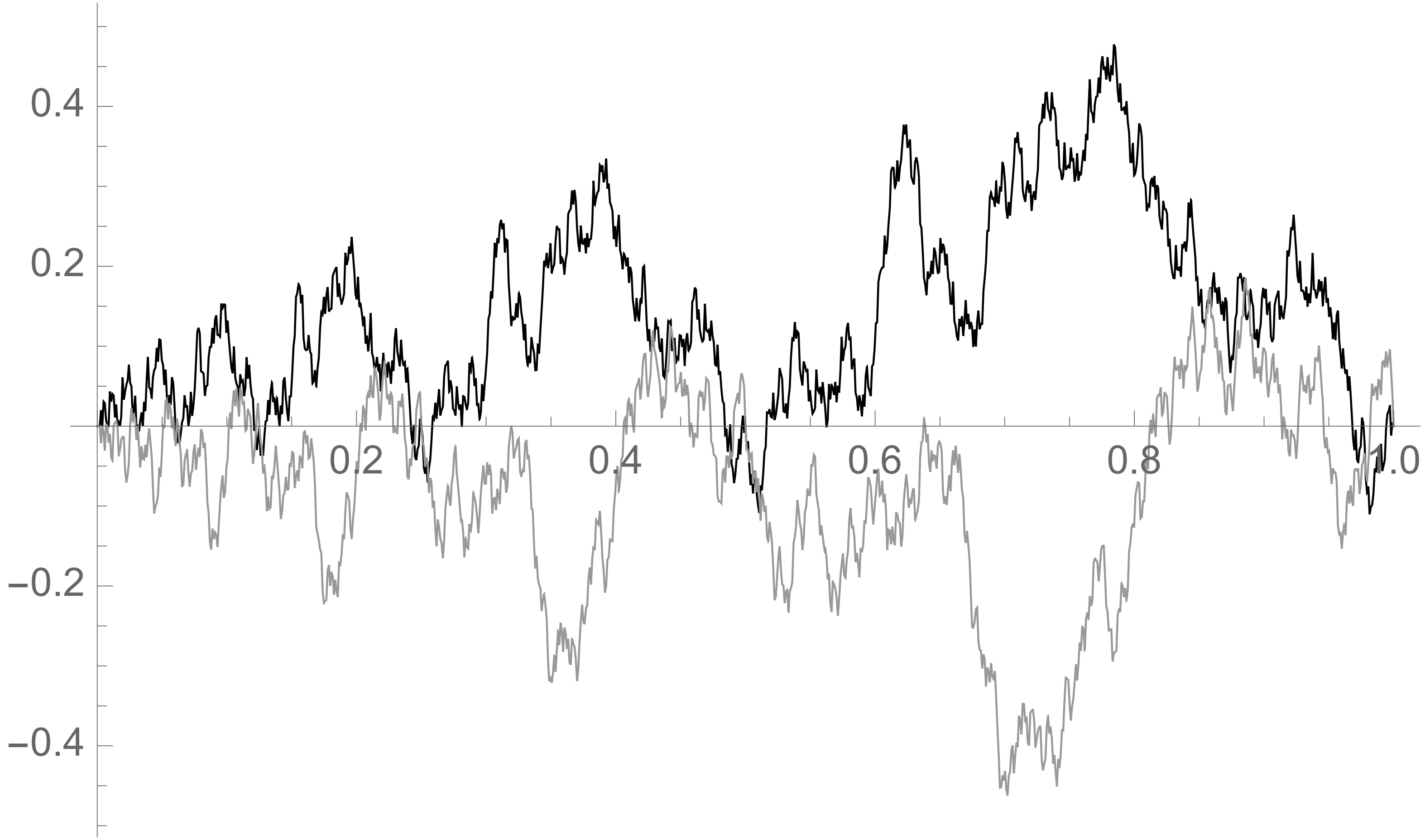
$$\frac{1}{n} \sum_{k=0}^{n-1} h\big(\alpha k \bmod 1\big) \longrightarrow \int_0^1 h(s) \, ds$$

for every Riemann integrable function h $\square$

Again, the class

$$\left\{ Y^{\alpha,f} \mid f \in \mathscr{R}[0,1] \right\}$$

is a vector space for each irrational α



The function $Y^{\alpha,f}$ for $\alpha = e$ (grey), $\alpha = 10e$ (black), and $f(t) := \sin 2\pi t$

2.4 Constructing functions with local quadratic variation

Recall that for options hedging as in Bick & Willinger (1994) we need functions Z satisfying

$$\langle Z\rangle(t) = \int_0^t \sigma(s, Z(s))^2 \, ds$$

First idea: apply a suitable time change to a function X with linear quadratic variation $\langle X\rangle(t) = t$.

However, the time-changed function will not necessarily admit a quadratic variation with respect to the original sequence of partitions, $(\mathbb{T}_n)$, but with respect to a new, time-changed sequence.

Instead, construct solutions to pathwise Itô differential equations of the form

$$dZ(t) = \sigma(t, Z(t))\, dX(t) + b(t, Z(t))\, dA(t)$$

where A is a continuous function of bounded variation (Mishura & A.S. 2016)

This can, e.g., be achieved by means of the Doss–Sussmann method combined with the following associativity property of the Föllmer integral (A.S. 2014):

$$\int_0^t \eta(s)\, d\Big(\int_0^s \xi(r)\, dX(r) \Big) = \int_0^t \eta(s)\xi(s)\, dX(s)$$

Conclusion

- Many financial problems can be formulated in a probability-free manner by means of pathwise Itô calculus, thus addressing the issue of model risk
- In a pathwise formulation, the actually required modeling assumptions become more transparent.
- Pathwise Itô calculus works not only for integrators that are sample paths of semimartingales but also for many fractal functions
- Pathwise Itô calculus is more elementary than standard stochastic calculus and thus a great means of teaching continuous-time finance

Thank you

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Bick, A. & Willinger, W. (1994), 'Dynamic spanning without probabilities', *Stochastic Process. Appl.* **50**(2), 349–374.

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