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Chapter 4

Some basic principles in interaction calculations

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Abstract

Some general aspects of the interactions taking place within a suspension can be understood by exploiting properties of the Stokes equations; these properties and their applications are described. Specifically, the way in which reversibility can be used to predict overall properties of a flow is explained by analysing several applications both informally and formally; Faxén's laws for the response of a particle to an ambient flow are examined to clarify common conceptual difficulties; and the use of lubrication theory to approximate interactions between close particles is developed carefully. In addition, the ways in which tensors can be used to summarize interaction results are covered, and the principles are illustrated whereby tensor relations can be simplified by appealing to geometrical symmetries and the character of tensor transformations.

Nomenclature

a	Radius of spherical or circular particle
$\mathbf{A}, \mathbf{B}$	Resistance tensors
$\mathbf{E}$	Rate-of-strain tensor
$\boldsymbol{f}$	Equivalent surface forces
$\boldsymbol{F}, F_i$	Force on particle
$\boldsymbol{g}$	Acceleration due to gravity
$\mathbf{G}$	Green's function for Stokes flow
G_{ijk}	Resistance tensor

h	height of lubrication gap
$K_\perp, K_\parallel, K_i$	Resistance coefficients
L	Length of lubrication gap
$\boldsymbol{L}$	Couple acting on particle
M	Mass of particle
$\boldsymbol{n}$	Unit normal vector
p	Pressure
$\boldsymbol{u}$	Fluid velocity
$\boldsymbol{u}^\infty$	Ambient fluid velocity
u, v, w	Velocity components
U, V, W	Scaled velocity components
$\boldsymbol{U}, \mathcal{U}$	Velocity on a boundary
$\boldsymbol{V}$	Velocity of a particle
t	Time
$\boldsymbol{t}$	Unit tangential vector
T	Period of motion
$\boldsymbol{x}$,$\boldsymbol{x}_0$	Position vectors
$\boldsymbol{x}_c$	Centroid of particle
γ	Shear rate
ϵ	Small nondimensional gap
$\boldsymbol{\epsilon}, \epsilon_{ijk}$	Alternating tensor
$\boldsymbol{\xi}$	Point on surface of particle
μ	Viscosity
$\boldsymbol{\sigma}$	Stress tensor
$\boldsymbol{\Sigma}$	Derivative of Green's function for Stokes flow
$\boldsymbol{\Omega}$	Angular velocity of particle
$\boldsymbol{\omega}$	Vorticity, angular velocity

1 Introduction

The Stokes equations for low-Reynolds-number flow are obtained from the Navier-Stokes equations by omitting the inertia terms, and are

$$\nabla p = \mu \nabla^2 \boldsymbol{u} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{u} = 0 \ , \tag{1}$$

where $\boldsymbol{u}$ is the velocity field of the fluid, p is the pressure and μ is the viscosity (Batchelor 1967; Happel & Brenner 1963; Kim & Karrila 1991). If the systems being considered consist of rigid walls and rigid particles, the boundary conditions are the 'no slip' ones: on all boundaries $\boldsymbol{u} = \boldsymbol{U}$, where $\boldsymbol{U}$ is the local motion of the boundary.

Since the Stokes equations are a set of linear equations, they are mathematically more tractable than the nonlinear Navier-Stokes equations. This means that it is possible to obtain more detailed information

about the interactions between suspended particles at low Reynolds numbers than it is at higher Reynolds numbers. The calculations are lengthy, however, and the large amount of daunting detail involved can make it difficult for a person starting in the field to follow the procedures being used. For this reason, any simplifications that are possible in particular cases should be exploited. There are several mathematical tools available to simplify calculations, although some of them require familiarity with the more advanced aspects of vector and tensor analysis; these tools are the subject of this chapter.

2 Stokes equations and reversibility

The property of reversibility states that a solution of the Stokes equations is valid, under appropriate conditions, both if time runs forward and if time runs backwards. By itself, reversibility implies interesting effects, but it gains greater significance when combined with other symmetry arguments, because then some surprisingly powerful statements can be deduced. For example, the fact that a spherical particle sedimenting near a vertical wall always maintains the same perpendicular distance from the wall can be demonstrated without solving any specific flow problem. In a similar way, reversibility predicts that a tumbling, nonspherical particle in shear flow will execute a periodic orbit and will not drift to any preferred orientation. The ease with which such predictions can be obtained contrasts strongly with the difficulty of their verification by detailed calculations of the forces acting on the particles; in particular cases the effects can appear to rely on the miraculous cancellation of unlikely terms.

A mathematical proof of reversibility is tedious to write out, so it is better to start with an informal discussion of the idea and its applications. The property of reversibility arises because the Stokes equations are linear and do not contain the time explicitly. The omission of the time derivative $\partial \boldsymbol{u}/\partial t$ from the Navier-Stokes equations has the consequence that time-dependent effects can enter a flow problem only through the boundary conditions, and through the evolution of particle positions. To put it another way, the flowfield $(\boldsymbol{u}, p)$ at any time t is determined completely by the positions and velocities of the boundaries at that instant, and it is unaffected by any considerations of history. For example, if an expanse of fluid contains a sphere moving with a velocity $\boldsymbol{V}$, then the flow around the sphere can be found without regard to how the sphere

obtained its position and velocity; the same flow will be found whether the sphere is moving at constant velocity $\boldsymbol{V}$ or whether it is undergoing an oscillatory motion and has the velocity $\boldsymbol{V}$ only at that instant. Consequently, if time is reversed, there is no change in the governing equations, and the system retraces its steps, moving in a valid Stokes flow.

Some visually compelling demonstrations of reversibility can be seen in the film *Low-Reynolds-number flows* that was made by G.I. Taylor in 1967*. One demonstration, for instance, uses very viscous oil in the annulus between two concentric vertical cylinders, the inner cylinder being able to rotate about its vertical axis. Some dye is injected into the oil and then the cylinder is rotated, causing the dye to smear out, or so it seems. When the motion is reversed, however, all the dye returns to its starting position (except for a little blurring due to molecular diffusion). A similar demonstration, using the same apparatus, places a rigid body in the oil, and again the inner cylinder is rotated. The body translates and rotates away from its initial configuration, but when the wall motion is reversed, the body returns to its starting point. Another demonstration shows a small mechanical fish vainly trying to swim by flapping a tail. At high Reynolds number, it moves forward, but at low Reynolds number, each time the tail reverses its motion, so does the fish.

The above demonstrations require reversibility alone; the next one combines reversibility with another symmetry possessed by the flow. Consider a sphere falling parallel to a vertical plane wall. The fact that the sphere falls at a constant distance from the wall is demonstrated in the following way. First one imagines that the sphere falls for a short time downwards. For the sake of setting up a contradiction, it is necessary to conjecture that it is *not* the case that the sphere stays at a constant distance from the wall, and that instead it moves away from the wall. Now consider what would happen if time were to run backwards. Clearly the sphere would retrace its path and move closer to the wall as it rose. Now observe that a second way to get the sphere to move upwards is to reverse gravity while leaving time running forward. Because the wall is a plane, the flow situation after gravity has been reversed is identical to the situation before (notice that this geometrical symmetry is independent of the reversibility just described). Accordingly, if the sphere is subjected to an upward force, then, by the starting conjecture, it will move away

* It is a pity, in view of the fact that only a small number of films exist of G.I. Taylor, that this one was made when he had a heavy cold.

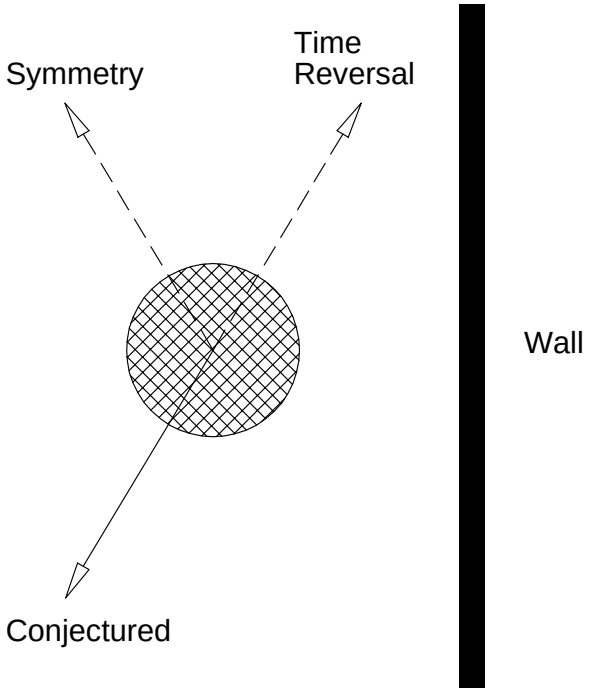

Figure 1. A sphere falling, or rising, next to a plane wall.

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from the wall as it rises, as shown in figure 1.

This is a contradiction: on the one hand, geometrical symmetry says that an upward-moving sphere will move away from the wall, while reversibility says that the sphere will move towards the wall. The conclusion is that it will neither move closer nor away, but stay at the same distance.

Reversibility is also important for periodic motion. Numerous periodic motions have been found in systems of particles moving in Stokes flow, and their existence can have a strong influence on calculations. If the system is scleronomic, meaning the external forces on the flow do not change with time, then a moving system of particles that passes through a configuration twice must be executing a periodic motion, and reversibility combined with another symmetry can prove this, as the tumbling of an ellipsoidal particle in a shear flow illustrates. The ellipsoid executes a closed periodic orbit, and this orbit is described by an orbital constant, whose value depends upon how the particle starts its motion. One might suppose that the ellipsoid would follow an orbit that is not exactly peri-

odic, and drift from one orbit to another, perhaps seeking out a preferred orbit. To fix a coordinate system, assume that the shear flow is given by $\boldsymbol{u} = \gamma z \boldsymbol{i}$, for some γ. If the ellipsoid were to alter its orbit and shift to one described by a larger (say) constant, then reversing time would cause it to return to the orbit with a lower constant. The flow, however, has a reflectional symmetry in the xy plane, and this reflection also reverses the shear flow (with time running forward) and therefore implies that the ellipsoid will still, in the reversed flow, increase its orbital constant. Again the contradiction can be resolved only by requiring that the particle remain in the same orbit always.

Quite large systems of particles can be trapped in periodic motion. Clusters of particles sedimenting under gravity in an infinite container have been studied by several authors (Hocking 1964; Caflisch *et al.* 1988; Tory *et al.* 1991; Tory & Kamel 1992; Golubitsky *et al.* 1991) and clusters of up to 6 particles were found to be capable of periodic motion. However, although reversibility and geometric symmetry were used during the proof of this result, by themselves they are not sufficient to guarantee periodicity.

The reader may have noticed that the examples given above share a common feature, viz. that reversibility was used in each case to say that something would not happen. In general terms, one might say that reversibility tends to prevent systems from evolving. It is clearly a powerful way of stopping anything interesting happening in the flow, and therefore it is useful to be clear on the limits of its stifling grip. The assumptions that are necessary for reversibility to be important can be examined by treating some simple examples in greater mathematical detail. One clarification is the fact that the principle of reversibility usually takes a stronger form than the one given in the opening statement of this section. For many systems, not only is the motion reversible, the displacements are the same whatever the rate at which they take place.

An example that captures the essence of G.I. Taylor's demonstrations in a simpler geometry is shown in figure 2. A box contains fluid and a suspended force-free particle, and is constructed so that one side can slide to create motion. The lower wall, $y = 0$, slides so that the point labelled X moves from $x = a$ at time $t = 0$, to $x = b$ at time $t = T$. At time t the point has position $x = X(t)$. The boundary conditions are then $\boldsymbol{u} = \boldsymbol{i}U(t) = \boldsymbol{i}dX/dt$ on $y = 0$ and $\boldsymbol{u} = 0$ on the other walls. The particle performs rigid-body motion, having velocity $\boldsymbol{V}$ and angular velocity $\boldsymbol{\Omega}$

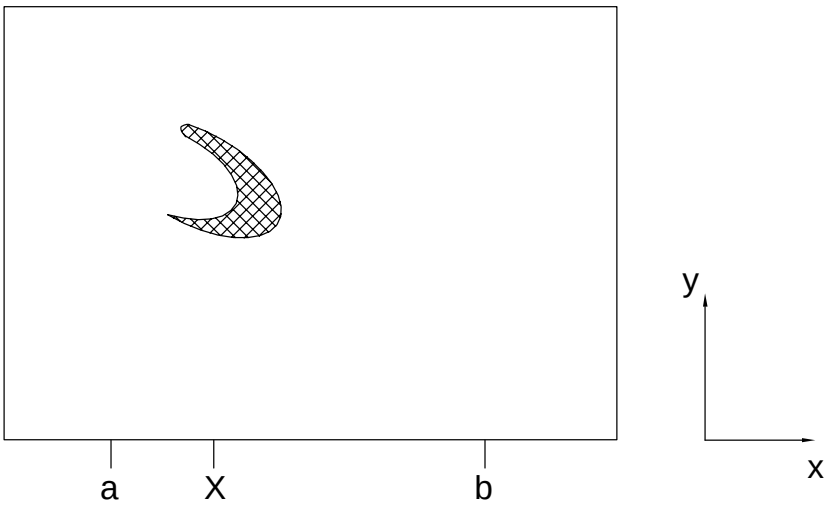

Figure 2. A simplified version of Taylor's demonstration.

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with respect to its centroid at $\boldsymbol{x}_c$. The boundary condition on its surface is then

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{\Omega} \times (\boldsymbol{x} - \boldsymbol{x}_c) \ , \tag{2}$$

and in addition $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ are chosen so that the total force $\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \ \mathrm{d}S$ on the particle is zero, as is the total moment $\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \times (\boldsymbol{x} - \boldsymbol{x}_c) \ \mathrm{d}S$. Here $\boldsymbol{\sigma}$ is the stress tensor. It will now be proved that, for the purpose of calculating the displacements of fluid particles, it does not matter how the point X gets from $x = a$ to $x = b$; in other words, for all possible functions $X(t)$, the fluid elements and the suspended particle will have undergone the same displacement, when the point X arrives at $x = b$.

The proof solves the flow problem at time t in two steps. First, let the wall have unit velocity, instead of its actual velocity $V(t)$, and let the corresponding velocity and pressure solutions be $\boldsymbol{u}^*(\boldsymbol{x})$ and $p^*(\boldsymbol{x})$, and the corresponding motion of the particle be given by $\boldsymbol{U}^*$ and $\boldsymbol{\Omega}^*$. Then the actual flow problem has solution $V(t)\boldsymbol{u}^*(\boldsymbol{x})$ and $V(t)p^*(\boldsymbol{x})$, and further the particle moves with $V(t)\boldsymbol{U}^*$ and $V(t)\boldsymbol{\Omega}^*$. The proof that the solution satisfies the governing equations depends on linearity:

$$\mu\nabla^2(V(t)\boldsymbol{u}^*) = \mu V(t)\nabla^2\boldsymbol{u}^* = \nabla(V(t)p^*) = V(t)\nabla p^* \ ,$$

and so the $V(t)$ cancels and $\mu\nabla^2\boldsymbol{u}^* = \nabla p^*$ by definition. The force-free

condition is also satisfied, because, using subscript notation,

$$\begin{aligned}\sigma_{ij} &= -p\delta_{ij} + \mu\left[\frac{\partial u_i}{x_j} + \frac{\partial u_j}{x_i}\right] \\ &= V(t)\left(-p^*\delta_{ij} + \mu\left[\frac{\partial u_i^*}{x_j} + \frac{\partial u_j^*}{x_i}\right]\right) = V(t)\sigma_{ij}^* \ .\end{aligned}$$

Since σ_{ij}^* integrates to zero force, so does σ_{ij}. The other boundary conditions are satisfied in a similar manner.

The displacement of a fluid element is calculated by considering what happens during a time δt. If a fluid element is at $\boldsymbol{x}_L(t)$ (the subscript is a reminder that $\boldsymbol{x}_L$ is a Lagrangian quantity), then its motion is given by

$$\frac{\mathrm{d}\boldsymbol{x}_L}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}_L(t), t) \ , \tag{3}$$

where $\boldsymbol{u}$ is the Eulerian velocity that obeys the Stokes equations. Therefore, in time δt, the displacement of the fluid element is given approximately by $\delta t\, \boldsymbol{u}(\boldsymbol{x}_L, t) = \delta t\, V(t)\boldsymbol{u}_L^*$, using the results established above. Now $\delta t\, V(t) = \delta x$, the displacement of the boundary. Thus all fluid displacements during the motion are proportional to the corresponding displacement of the boundary, implying that the total displacement of any particular element is proportional to $\int_0^T V(t)dt = X(T) - X(0) = b - a$, whatever the function $X(t)$. This can be seen in the demonstrations in Taylor's film, where the fact that the cylinder is turned in a variable way by hand is not important, so long as it is stopped at the same place each time. In addition, when the handle is returned to its starting position, only the overall displacement is important, meaning that each fluid element returns to its starting position when the handle does.

The above calculations were for systems with rigid particles, but similar considerations show that systems with free surfaces may also be candidates for reversibility. If a fluid contains a free surface or an interior void, one idealization is to suppose the surface to be stress free, i.e. $\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$, where $\boldsymbol{n}$ is a unit normal to the surface. For this case, the governing equations again admit a solution proportional to the instantaneous velocity of the driving boundary element. This is because $\boldsymbol{\sigma} \cdot \boldsymbol{n} = V(t)\boldsymbol{\sigma}^* \cdot \boldsymbol{n} = 0$. Another common approximation is to suppose that surface tension is so high that a void has a permanent spherical shape. In this case the appropriate boundary conditions are $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{V} \cdot \boldsymbol{n}$, where $\boldsymbol{V}$ is the rigid-body velocity of the spherical void (determined as

part of the solution), together with $\boldsymbol{t} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ where $\boldsymbol{t}$ is a unit tangent vector to the surface. The equations and boundary conditions are again linear and scale with the boundary velocity.

The strong form of reversibility applies to systems in which the Stokes equations contain a single time scale, imposed by the boundaries through the boundary conditions. Such cases will arise whenever a problem is simplified to a single effect. As soon as other time scales enter the system, reversibility is weakened. Consider allowing the particle in figure 2 to sediment because of gravity while the wall of the box is moved. In this case, the force-free boundary condition on the particle is replaced by

$$\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = M\boldsymbol{g} \ , \tag{4}$$

where M is the mass of the particle and $\boldsymbol{g}$ is the acceleration due to gravity. The problem no longer scales with the velocity of the boundary, because if $\boldsymbol{u}^*$ is defined as before, the new boundary condition becomes

$$\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = V(t) \int \boldsymbol{\sigma}^* \cdot \boldsymbol{n} \, \mathrm{d}S = V(t) M\boldsymbol{g} \ .$$

This makes sense physically, because the sedimentation rate of the particle is independent of the motion of the wall, and the particle will move different amounts during an experiment, depending upon whether the wall slides slowly or quickly. Thus, only a weakened reversibility argument could be constructed for this system, by tying the speed of the wall to the sedimentation rate, which is too restricted to find an application. Similarly, turning to the system containing a free surface, one could impose a more realistic boundary condition containing a finite amount of surface tension. This would again destroy the scaling of the solution with wall velocity.

In the context of rigid particles, the major way in which reversibility is destroyed in theoretical investigations is through departures from Stokes flow. The previous examples have considered different boundary conditions, while the governing equations have remained unchanged, but there are many ways in which the governing equations, themselves, can change. For example, the effects of small amounts of inertia have been investigated by a large number of authors, as have the effects of non-Newtonian terms in the stress-strain relation. Calculations based on these idea are reviewed in Leal (1980). More generally, all systems

containing very small particles will be subject to physical processes that modify the governing equations and destroy reversibility, such as electric charge effects, Brownian motion, and so on.

3 Faxén laws

Faxén laws are exact mathematical statements about the response of a particle to an ambient flow. They are used, however, mostly in the approximate calculation of interactions between particles, and this ambivalence can be a cause of confusion. Suppose that a region of fluid contains an ambient flow field $\boldsymbol{u}^\infty(\boldsymbol{x})$, and suppose that a rigid spherical particle is introduced into this field at a point $\boldsymbol{x} = \boldsymbol{x}_0$. The force $\boldsymbol{F}$ and couple $\boldsymbol{L}$ experienced by the sphere, which is stationary, are given exactly by

$$\boldsymbol{F} = 6\pi a\mu \left[\boldsymbol{u}^\infty(\boldsymbol{x}_0) + \tfrac{1}{6}a^2\nabla^2\boldsymbol{u}^\infty(\boldsymbol{x}_0)\right] \ , \tag{5}$$

and

$$\boldsymbol{L} = 8\pi a^2\mu\nabla \times \boldsymbol{u}^\infty(\boldsymbol{x}_0) \ . \tag{6}$$

Laws for other quantities are also known. It is certainly striking, on first acquaintance, to see that only the local velocity and its first and second derivatives are important, no matter how complicated the ambient flow may be. As Kim & Karrila (1991) point out, the interpretation can be simplified further by noting that $\nabla^2\boldsymbol{u}^\infty$ is proportional to the ambient pressure gradient. It is natural to contrast this simplicity with the lengthy calculations in the literature of the motion of a particle near a wall or a second particle, and to wonder how it can be exact. Students sometimes draw a diagram of a sphere in a maelstrom of streamlines, and ask whether Faxén laws applies to this situation.

An understanding of whether Faxén laws are exact or approximate can be obtained by reviewing a proof of them, here the proof that was given in the appendix to Batchelor (1972). Any solution of the Stokes equations can be written in the form of surface integrals over the boundaries of the region (Kim & Karrila 1991, section 2.4.2).

$$\boldsymbol{u}(\boldsymbol{x}) = \int [\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\mathsf{G}(\boldsymbol{x}-\boldsymbol{\xi}) + \boldsymbol{n}\cdot\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi})\cdot\boldsymbol{u}]\,\mathrm{d}S(\boldsymbol{\xi}) \ , \tag{7}$$

where $\boldsymbol{n}$ is a normal pointing into the fluid, G is a Green's function and $\boldsymbol{\Sigma}$ is its derivative. The first term in the integrand contains $\boldsymbol{n}\cdot\boldsymbol{\sigma}\ \mathrm{d}S$, which is the force exerted *on* the fluid *by* the surface element $\mathrm{d}S$ at $\boldsymbol{\xi}$.

Then $\mathsf{G}(\boldsymbol{x}-\boldsymbol{\xi})\cdot(\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}S)$ is the velocity field produced at $\boldsymbol{x}$ by that force. The second term $\boldsymbol{n}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{u}\,\mathrm{d}S$ corresponds to a 'double layer', and has a similar interpretation. The solution (7) can be interpreted by saying that the flow outside a rigid particle cannot tell the difference between a physical boundary and an appropriate distribution of point forces and double layers. Faxén's theorem is derived by adding such a distribution of forces to a region of fluid so as to represent the presence of a particle. Suppose that there is a point $\boldsymbol{x}_0$ in the fluid such that all boundaries are at least a distance R away, and suppose these distant boundaries have produced a flow field $\boldsymbol{u}^\infty$ near $\boldsymbol{x}_0$. Finally, suppose that forces $\boldsymbol{f}$ have been found that emulate the presence of an additional sphere. In general, double layers are also needed, but Kim & Karrila (1991, 2.4.3) show that the force layer is sufficient in the case of a rigid particle. Then the velocity field near $\boldsymbol{x}_0$ is

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^\infty + \int \boldsymbol{f}(\boldsymbol{\xi})\cdot\mathsf{G}(\boldsymbol{x}-\boldsymbol{\xi})\,\mathrm{d}S(\boldsymbol{\xi})\ . \tag{8}$$

Integrating over all points $\boldsymbol{x}$ on the surface of the sphere, where $\boldsymbol{u}=0$, proves the theorem (Batchelor 1972).

For the present discussion, the important question is how the forces $\boldsymbol{f}$ modify $\boldsymbol{u}^\infty$. It was postulated that $\boldsymbol{u}^\infty$ was produced by distant boundaries, at least a distance R away. Simple examples of the interaction of a point force with a boundary are well known. The flow produced by a point force at a distance R from a rigid plane is equal to the sum of the flow produced by the force and the flow produced by an equal point force on the other side of the plane, a distance R from it, and some weaker effects (Lorentz 1896). Similarly, a point force outside a sphere produces 'reflections' inside the sphere (Fuentes, Kim & Jeffrey 1988). Since $\boldsymbol{u}^\infty$ includes all of the effects of the distant boundaries, it will have to be corrected for the additional forces introduced by the reflections. The order of magnitude of these corrections can be estimated by saying that the images have the same magnitude as the forces $\boldsymbol{f}$, and they are at least a distance R away. Since a point force produces a velocity field that decays as the reciprocal of distance, the disturbance to $\boldsymbol{u}^\infty$ will be $O(R^{-1})$. This discussion shows that another way to write Faxén's law is

$$\boldsymbol{F} = 6\pi a\mu\left[\boldsymbol{u}^\infty(\boldsymbol{x}_0) + \tfrac{1}{6}a^2\nabla^2\boldsymbol{u}^\infty(\boldsymbol{x}_0)\right] + O(R^{-1})\ . \tag{9}$$

The usual statement of the law assumes that the limit $R\to\infty$ has been taken.

Some of the usual questions about Faxén laws can now be answered. The first question asks 'Is it really exact?'. The answer is 'Yes', provided the limit $R \to \infty$ is accepted. This limit is often taken in fluid mechanics, and although it can cause difficulties, for example uniform flow around a cylinder in two dimensions, it is usually accepted if $\boldsymbol{u}^\infty$ is simple. The second question asks 'Does it apply to flows with large curvature?'. Large curvature means that the values of $\nabla \boldsymbol{u}^\infty$ and higher derivatives are large, in order for the streamlines of $\boldsymbol{u}^\infty$ to be highly curved. Here the answer is 'Technically yes, but in practice no'. It is technically yes, because it is possible mathematically to imagine a flow with large curvature produced by very distant boundaries. It is no in practice, because the person asking the question is almost certainly thinking of a situation in which the curvature is caused by boundaries fairly close to the particle. For example, the particle is in a curving pipe or near other particles. Thus the flow is probably not caused by large forces far away, but by ordinary forces that are close. In this case the questioner is really refusing to ignore the reflections from the boundary, in other words, refusing to accept the premise of the theorem.

4 Lubrication theory and close particles

The interactions between two nearly touching particles can often be calculated using *lubrication theory*. This 'theory' is actually a set of approximations that allow the Navier-Stokes equations to be simplified to a form in which they are more easily solved. In order for the approximations to be valid, the particles must be nearly touching, and in addition their relative motion must cause the fluid in the gap between them to be highly sheared. For example, if two close particles are in the process of moving past each other, then lubrication theory can be applied, but if they are sedimenting side-by-side with no relative motion between them, it cannot. Although the name lubrication theory is a reminder that the approximations were first worked out in the analysis of flows in lubricated bearings, the theory in the low-Reynolds-number context is not identical to its engineering namesake; some of the ways in which the approximations are developed in Stokes flow give the application a distinctive slant.

The aspects of the theory that will be explored are the definition of the gap between the particles, the handling of the edge of the gap, and the role of the fluid far from the gap. It is important to discuss these points

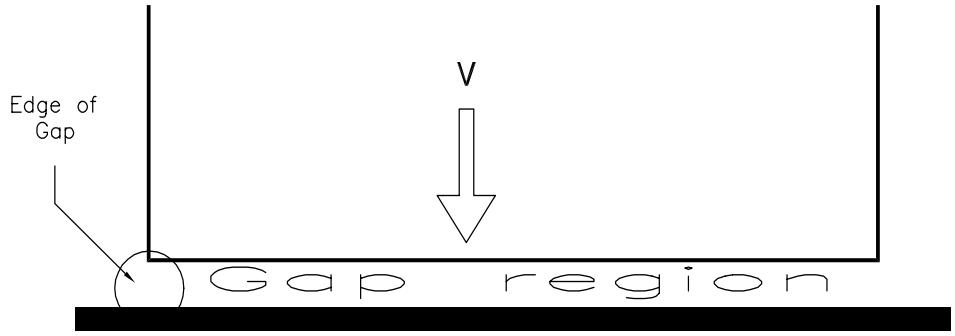

Figure 3. A plate approaching a wall.

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because the justification for the theory is usually informal; it is developed initially for a situation in which the approximations have a strong intuitive appeal, or for a situation in which they can be proved correct. Subsequently, the approximations are applied to situations in which the justification is less straightforward, and there is a danger that future calculations will continue to use them when they have really ceased to apply. The discussion here follows a pattern similar to that just described: it starts with an example in which the justification for the approximations is easy to accept and then proceeds to greater mathematical elaboration.

A simple two-dimensional flow serves as a starting point: consider a flat plate approaching a plane at speed V. The fluid between the plate and the plane occupies the rectangular region labelled in figure 3 as the gap region. The region has length $2L$ and height h, and h decreases with time at the rate $dh/dt = -V$. In order for the plate to move, the fluid in the gap must be squeezed out. During time δt, the volume of the gap decreases by $2LV\delta t$ (for unit length in the third dimension), and since this equals the flux of fluid at the two edges of the gap, the fluid must be leaving with an average velocity LV/h. The only force available to drive this flow is a pressure gradient in the gap, implying that the pressure must be highest in the centre of the gap in order to create the required gradient. This pressure is calculated approximately by considering the order of magnitude of each term in the Stokes equations. Let x and y measure distance respectively along the gap and across it, and let the

non-dimensionalized velocity components be given by $\boldsymbol{u} = V(u, v)$. If the pressure is given by $\mu V p$, the equations are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \ , \tag{10}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial p}{\partial y} \ , \tag{11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ . \tag{12}$$

The no-slip boundary conditions require that

$$u(x,0) = u(x,h) = 0 \quad \text{and} \quad v(x,0) = 0 \quad \text{and} \quad v(x,h) = -1 \ .$$

Since, from the boundary conditions, v goes from -1 to 0 in a distance h, the term $\partial v/\partial y$ will be approximately $-1/h$, and then, from the continuity equation (12), this implies that $\partial u/\partial x \approx 1/h$. This in turn implies $u \approx x/h$, which is consistent with our global estimate above. Since u grows with x while v does not, it must be that $u \gg v$ for $x \gg h$, i.e. away from the centre of the gap. So the terms in (11) are much less than those in (10), and (11) can be neglected. Further, $\partial p/\partial y$ can be set to zero in comparison with $\partial p/\partial x$, meaning that p is approximately independent of y. The assumption that pressure is approximately constant across a thin layer of fluid is also used in boundary-layer theory. In equation (10), the derivative $\partial^2 u/\partial y^2$ must be of the order of Vx/h^3, since u changes from 0 at the walls to Vx/h in the flow. The estimate $u \approx Vx/h$ also implies $\partial^2 u/\partial x^2 \approx 0$, so clearly $\partial^2 u/\partial y^2 \gg \partial^2 u/\partial x^2$, and therefore (10) can be approximated by

$$\frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{d}p}{\mathrm{d}x} \ , \tag{13}$$

the total derivative of p showing that it depends only on x. Integrating (13) gives $u = \frac{1}{2}p'y(y-h)$, where $p' = \mathrm{d}p/\mathrm{d}x$, and substituting into (12) and integrating gives $v = \frac{1}{12}p''y^2(3h-2y)$. At this stage the pressure is still unknown. The expression for v satisfies the boundary condition on $y = 0$, but it can only satisfy $v = -1$ on $y = h$ if p satisfies the equation

$$\frac{h^3}{12}\frac{d^2 p}{dx^2} = -1 \ . \tag{14}$$

Therefore $p = -6x^2/h^3 + Ax + B$. The calculation is completed by applying boundary conditions on p. By the symmetry of the problem, $A = 0$, but what about B, the pressure at the centre of the gap?

As the plate approaches the plane, the fluid from the gap region will escape into the surrounding fluid, spreading out until it reaches the ambient pressure. Depending upon the Reynolds number of the flow outside the gap, this could even be a jet-like flow. The problem of how the fluid slows down on leaving the gap is a difficult calculation and it is better avoided if possible. Consequently, it is assumed that as soon as the fluid reaches $x = \pm L$, the pressure equals the pressure outside the gap (which is zero). This gives $p = 6(L^2 - x^2)/h^3$, and a non-dimensionalized force of $2\int_0^L p \, \mathrm{d}x = 8L^3/h^3$. The last assumption is in fact the specification of the edge of the gap (which was left vague above) and the reader's acceptance of it depends to some extend on mathematical outlook. The clear-cut geometry suggests strongly that the pressure will reach the ambient one within a distance $O(h)$ from the edge, that is, within the circle labelled 'edge of gap' on figure 3. This will induce an error $O(h/L)$, which is of the same order as the approximations made in obtaining (13). It is possible to contrive flow conditions outside the gap that would invalidate the assumptions about the edge of the gap, but it is not a serious worry and the approximations above are well established.

The above example is important in engineering lubrication theory, but for sedimentation studies and other particulate interaction problems, the geometry is only generally relevant. Keeping with the simplification of two-dimensions, consider a cylinder approaching a plane, again with velocity V, as shown in figure 4. From looking at the figure, one can be convinced that there is a gap region between the cylinder and plane, although it is no longer possible to point to the edge of the gap with the confidence felt in the first example. One proceeds to analyse the flow in the gap, in the hope that everything will turn out all right in the end.

Let the radius of the cylinder be a and let the gap be h at its minimum. The velocities and pressure are again $V(u, v)$ and $\mu V p$ and the approximations leading to (12) and (13) still apply. The boundary conditions can be simplified by expanding the expression for the cylinder surface for small values of x. Thus

$$y = a + h - \sqrt{a^2 - x^2} = h + \frac{x^2}{2a} + O(x^4) = H + O(x^4) \ , \qquad (15)$$

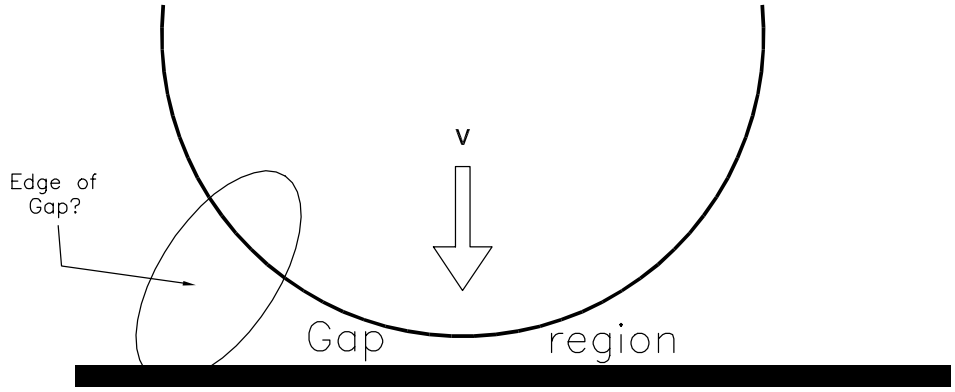

Figure 4. A cylinder approaching a wall.

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where H has the obvious definition. The omission of the terms $O(x^4)$ can be justified formally using a scaling argument, but the formal treatment is deferred until later. The solutions of (12) and (13) subject to the conditions $u(x,0) = u(x,H) = v(x,0) = 0$ and $v(x,H) = -1$ are

$$p = \frac{6a}{H^2} + A\left(\frac{3x}{H} + \frac{2hx}{H^2} + 3\sqrt{\frac{2a}{h}}\arctan\frac{x}{\sqrt{2ah}}\right) + B\ , \tag{16}$$

$$u = \frac{1}{2}\frac{dp}{dx}y(y-H)\ , \tag{17}$$

$$v = \frac{1}{12}p''y^2(3H-2y) + \frac{p'xy^2}{4a}\ . \tag{18}$$

As in the previous case, $A = 0$ by the symmetry of the problem, and if it is provisionally assumed that there is a point $x = L$ where the gap ends and $p = 0$, one obtains $B = -24a^3(L^2+2ah)^{-2}$. Now some interesting things can be observed. Unlike the first example, the value of B is not so critical here. For example, the pressure at the centre of the gap is $6a/h^2 + B$, and provided $L \gg h$, the contribution of B is unimportant. Further, the force on the cylinder is

$$F \approx 2\int_0^L p\ \mathrm{d}x \approx 3\sqrt{2}\pi\left(\frac{h}{a}\right)^{-\frac{3}{2}}\left[1 + \frac{8\sqrt{2}}{3\pi}\left(\frac{ah}{L^2}\right)^{\frac{3}{2}}\right]\ , \tag{19}$$

and this expression depends only weakly on L. The influence of the edge of the gap can be suppressed completely by taking the limit $L \to \infty$, and

in some presentations, this limit is taken early in the treatment, so that the edge of the gap never appears.

The lubrication result (19) can be compared with the exact result obtained by Jeffrey & Onishi (1981). When their solution is expanded for small h, it becomes

$$F = 3\sqrt{2}\pi \left(\frac{h}{a}\right)^{-\frac{3}{2}} + \frac{63\sqrt{2}}{20}\left(\frac{h}{a}\right)^{-\frac{1}{2}} + O\left(\left(\frac{h}{a}\right)^{\frac{1}{2}}\right) . \qquad (20)$$

Both (19) and (20) contain the same leading term, but (20) contains a second singular term. Numerically, both terms are important: if $h = 0.1$, then the force according to the exact solution is 148.31, whereas one term of (15) gives 134.2 and two terms give 148.25. So the comparison provides reassurance on one point, but raises another. In particle geometries, the edge of the gap can be avoided to leading order, but lubrication theory must be extended to higher order to capture all of the important terms. Can the edge of the gap be ignored at higher order? This question has been investigated in three-dimensional flows.

The change from two dimensions to three dimensions weakens the interactions between close particles, and as a result the edge of the gap (wherever it is) threatens to stay in the calculations. In order to justify lubrication theory in this case, O'Neill & Stewartson (1967) constructed a complete matched-asymptotic-expansion analysis of a specific problem. It is worth quoting some of the introductory remarks of their paper concerning the status of lubrication theory. They wrote '... although it [the theory of lubrication] has been extensively developed for many years ... it suffers from a number of drawbacks. First, it is difficult to perform experiments ... Secondly, the theory has developed on an *ad hoc* basis ... Furthermore, in certain types of problems, ... what is known about the actual flow properties makes one sceptical of the *sufficiency* of lubrication theory.' (The italics are the authors'.) Thus although many writers use the terms 'well established' and 'widely accepted' interchangeably, O'Neill & Stewartson apparently saw lubrication theory as being only the latter, and not yet the former.

Their lengthy analysis is given here only in outline, just enough to exhibit their main achievements, which are a formal scheme that allows the approximations to be improved, and a proper treatment of the edge of the gap. If the sphere has radius a, nondimensional cylindrical coordinates are (ar, θ, az), where the z axis is perpendicular to the plane.

Letting $\mathcal{U}$ be the velocity of the sphere along the x axis, the velocity can be nondimensionalize and the dependence on θ removed by writing the velocity in the form $\boldsymbol{u} = \mathcal{U}(u\cos\theta, v\sin\theta, w\cos\theta)$ and the pressure as $\mu\mathcal{U}p\cos\theta/a$. The gap is again h, and the small parameter ϵ is defined to be $\epsilon = h/a$. The centre of the sphere is then at $z = 1 + \epsilon$. The formal recognition that $\partial u/\partial z \gg \partial u/\partial r$ is made by introducing new variables $R = r/\epsilon^{1/2}$ and $Z = z/\epsilon$, because (to be brief) at the centre of the gap, z runs from 0 to ϵ and Z runs from 0 to 1. The derivatives can now be compared by converting them to R and Z.

$$\partial/\partial z = \epsilon^{-1}\partial/\partial Z \gg \partial/\partial r = \epsilon^{-1/2}\partial/\partial R \ .$$

Similarly, the assertion $u \gg w$ is formalized by putting $(u, v, w) = (\epsilon^{-1/2}U, V, W)$. The Stokes equations can be approximated by keeping the leading terms in ϵ.

$$\frac{\partial P}{\partial R} = \frac{\partial^2 U}{\partial Z^2} \ , \qquad -\frac{P}{R} = \frac{\partial^2 V}{\partial Z^2} \ , \tag{21, 22}$$

$$\frac{\partial P}{\partial Z} = 0 \ , \qquad \frac{\partial U}{\partial R} + \frac{U+V}{R} + \frac{\partial W}{\partial Z} = 0 \ . \tag{23, 24}$$

The approximation of the upper surface of the gap is similar to the two-dimensional case, and is $Z = H = 1 + \frac{1}{2}R^2$. Then $U = -V = 1$ on $Z + H$ and $W = 0$. The solution is

$$P = 6R/5H^2 \ , \tag{25}$$

$$U = \frac{6 - 9R^2}{10H^3}Z^2 + \frac{2 + 7R^2}{5H^2}Z \ , \tag{26}$$

$$V = -\frac{3}{5H^2}Z^2 - \frac{2}{5H}Z \ , \tag{27}$$

$$W = \frac{8R - 2R^3}{5H^4}Z^3 + \frac{2R^3 - 7R}{5H^3}Z^2 \ . \tag{28}$$

The non-dimensionalized force on the plane $F_x\boldsymbol{i}$ is given, after integrating with respect to θ, by an integral containing the edge of the gap region R_G.

$$F_x = \int_0^{R_G} \left(\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z}\right)_{Z=0} R \ \mathrm{d}R = \frac{8}{15}\log(1 + \tfrac{1}{2}R_G^2) - \frac{2}{5} + \frac{4/5}{2 + R_G^2} \ .$$

The edge of the gap appears in the expression for the force in a way that cannot be removed by taking the limit $R_G \to \infty$, and this is the feature that this example was chosen to show. The bolder calculators now continue by replacing R_G by its unstretched counterpart $r_G^2 = \epsilon R_G^2$. To leading order in ϵ, the force becomes

$$F_x = \frac{8}{15}[\log(r_G^2) - \log\epsilon - \log 2] - \frac{2}{5} + O(\epsilon) \ .$$

They would continue by arguing that the value of r_G can be at most the radius of the sphere, and so is a number O(1). This leads immediately to the estimate

$$F_x = \tfrac{8}{15}\log\epsilon^{-1} + O(1) \ . \tag{29}$$

The advantage of this strategy is that the main result, namely the singular term in the force law, is arrived at quickly. The disadvantage is that the O(1) constant cannot be obtained, and the argument is not satisfying to the more mathematically fastidious. An early use of this approach, in a heat-conduction context, was described in Keller (1963).

In order to obtain a stronger demonstration that the main result really is independent of the outer boundary, O'Neill & Stewartson solved for the flow around a sphere translating while touching the plane (low Reynolds number everywhere in the flow). The simplification in the geometry is sufficient to express the solution in terms of one ordinary differential equation, although it is still too lengthy to reproduce here. When the force on the plane is calculated from this solution, it is infinite. Consequently, the lubrication region is invoked again, but now it is excluded from the problem outside the gap. Thus the force integral is computed by integrating over all $r > r_G$, and is found to be

$$F_x = \tfrac{16}{15}\log(2/r_G) + K \ ,$$

where K is a known integral that must be evaluated numerically. This calculation shows two things. If the integration were carried until $r = 0$, the result would be infinite, but if it is added to the inner force contribution, the two terms in $\log r_G$ cancel, leaving a finite result. Thus O'Neill & Stewartson finally concluded that

$$F_x = \tfrac{8}{15}\log\epsilon^{-1} + 0.9543 \ . \tag{30}$$

The logarithmic term can be confidently ascribed to the gap solution, but the constant term contains contributions from each region, and so cannot be ascribed to either one separately.

On of the attractions of lubrication theory is the fact that it can yield important information about interactions from a relatively simple analysis of only one part of the flow. If the edge of the gap cannot be removed from the calculations, lubrication theory loses some of its appeal. Thus the importance of the O'Neill & Stewartson result lies in the fact that it provides a foundation for other lubrication calculations to draw on. Even if the edge of the gap remains in the result of a gap calculation, it can be removed by postulating that a full analysis would find a cancelling term in the solution outside.

In a number of publications, lubrication theory has been pushed even further. Thus the approximations above have been carried to higher order in Jeffrey (1989), where it is shown that eventually some global properties of the flow must be taken into account in order to continue; by this stage, however, the interactions are extremely weak, and of only specialized interest. In another direction, the requirement of rigid surfaces has been relaxed and the approach of two drops considered (Davis *et al.* 1989). It is in the nature of approximations that they will be applied to weaker and weaker interactions until they break down. As the borders of applicability are approached, from various directions, the more formal treatment described here, although more cumbersome than the intuitive approach, becomes increasingly important, because it will show when the region of applicability of the approximations has been left. In the context of heat conduction, one analysis of the accuracy of extensions such as those just discussed has been published in Kocabiyik & Jeffrey (1994), but none in fluid flow.

The important conclusions remain, however, that the singular behaviour of the force between close particles is determined by the gap between them and their relative motion. It is not possible to avoid the forces becoming singular within the assumptions of Stokes flow and rigid particles; only by changing the boundary conditions can finite interaction forces be obtained. Another important conclusion is the fact that clusters of close particles interact to leading order in a pairwise fashion. Thus the total force on one particle is the sum of lubrication interactions with its nearly touching neighbours.

O'Neill & Stewartson pointed out the dearth of experimental verification of lubrication theory that pertained in 1967. Since then, there have been some improved experimental tests of the predictions of lubrication theory. For a cylinder approaching a plane, Trahan & Hussey (1985)

found good agreement with experiment. The motion of one sphere approaching a much larger one has been followed very accurately by Lecoq *et al.* (1995) and excellent agreement found with the theory.

5 Review of vectors and tensors

The interactions between particles are essentially vectorial in nature, but the difficulties of accounting for this can be made worse by some simple misconceptions about vectors and tensors. Consequently, this section reviews vectors and tensors, with an emphasis on those ideas that are not commonly treated in introductory courses, and yet are important in Stokes flow. It is assumed that the reader already has some familiarity with vectors and tensors, whether written in bold-face or subscript notation. For those not familiar with tensors, the books by Aris (1962) and Borisenko & Tarapov (1968) are two of many good introductions.

It is a common misconception that a vector is defined to be a quantity that has a length and a direction. Most textbooks on mechanics introduce vectors in this way, because such a definition is easily understood, it corresponds to a simple mental picture, and it allows beginning students to cover all of the basic material of mechanics. (Davis 1986; Morse & Feshbach 1953, p 9). However, in order to simplify vectorial equations in the context of Stokes flow, this definition must be modified. The more complete definition is that a vector describes a quantity that has a length and a direction *in a way that is not limited to any one coordinate system.* This definition is made mathematical by considering explicitly how the description of a vectorial quantity changes as the coordinates change, and it then becomes necessary to divide vectors into polar and axial vectors (Morse & Feshbach 1953, p10).

Consider two coordinate systems, for simplicity, Cartesian systems. Position $\boldsymbol{x}$ is given in the first system by components (x_1, x_2, x_3) and in the second by components (x_1', x_2', x_3'). The law relating these two sets of components is $x_i' = a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$, where the Einstein summation convention is written out (just this once). It should be noted that the quantity a_{ij} is not a tensor; it is another misconception that anything with two subscripts is a tensor. A vector $\boldsymbol{f}$ is now defined to be a collection of 3 components, either (f_1, f_2, f_3) in the first system or (f_1', f_2', f_3') in the second, such that the f_i are functions of $\boldsymbol{x}$ and such that $f_i' = a_{ij}f_j$. In words, the components of $\boldsymbol{f}$ transform according to the same law as that for position.

The need for axial vectors is demonstrated by considering a rotating rigid body in a very simple case: a rigid body rotating around the z axis with vector angular velocity $\boldsymbol{\omega} = \Omega \boldsymbol{k}$. Consider velocities in the XY plane. The velocity (v_x, v_y) at any point (x, y) is given by $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, meaning its components are given by

$$v_x = -\Omega y \ , \qquad v_y = \Omega x \ . \tag{31, 32}$$

First consider the transformation to a dashed coordinate system defined by $x' = y$, $y' = -x$ and $z' = z$, which is a rotation about the z axis. The matrix $a_{ij}^{(1)}$ for this transformation is

$$a^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{33}$$

The matrix $a^{(1)}$ can be used to transform both $\boldsymbol{\omega}$, which remains $\Omega \boldsymbol{k}$, and $\boldsymbol{v}$ whose components in the new coordinates become $v_x' = v_y$ and $v_y' = -v_x$. Now consider the transformation to another new, doubly dashed, coordinate system defined by $x'' = x$, $y'' = y$ and $z'' = -z$. The reader will notice that the new system is left handed. The matrix $a_{ij}^{(2)}$ for this transformation is

$$a^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ . \tag{34}$$

Clearly, this transformation causes the z component of any vector to change sign, since anything pointing along the old positive z axis must now point along the negative z'' axis. Suppose that this applies to the quantity $\boldsymbol{\omega}$. The transformation $\omega_z' = -\omega_z = -\Omega$ would lead to a difficulty, because the velocity components for $\boldsymbol{v}$ would become

$$v_x' = -\omega_z' y' = \Omega y \quad \text{ and } \quad v_y' = \omega_z' x' = -\Omega x \ .$$

Thus the x and y velocities are now the negative of what they were before, although those axes were unchanged. The only way to correct this is to introduce a negative sign somewhere in the chain of definitions, and the standard place is the transformation law for $\boldsymbol{\omega}$. Thus the equation $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ is kept intact (as is the definition of cross product in terms of components) as is the fact that $\boldsymbol{v}$ and $\boldsymbol{r}$ transform according to the

vector law. Now, however, $\boldsymbol{\omega}$ transforms according to a pseudovector law, namely $\omega_i' = a_{ij}^{(1)}\omega_j$, but $\omega_i'' = -a_{ij}^{(2)}\omega_j$. This is the difference between ordinary, or polar, vectors such as velocity, force and position, and axial vectors such as angular velocity, and angular momentum.

The difference between vector types must be respected when constructing equations to describe fluid flows, because in any equation each term must be the same type. Consider, as an example, the Navier-Stokes equation in the form (Batchelor 1967, eq 5.1.1),

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega} = \boldsymbol{F} - \nabla(p/\rho + \tfrac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}) + \nu\nabla^2\boldsymbol{u} \ . \tag{35}$$

Here, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity, and an axial vector. Each term in the equation is a polar vector, and in particular $\boldsymbol{u} \times \boldsymbol{\omega}$ is a polar vector because an axial vector crossed with a polar gives a polar. A similar analysis applies to the vorticity equation (Batchelor 1967, eq 5.2.2)

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} \ . \tag{36}$$

This contains axial vectors, and in particular the term $\boldsymbol{\omega} \cdot \nabla\boldsymbol{u}$ is axial because an axial vector dotted with a polar gives an axial.

If subscript notation is used instead of bold face, then the alternating tensor ϵ_{ijk}, which appears in the subscript equivalent of cross product, is a pseudotensor. Thus $v_i = \epsilon_{ijk}\omega_j x_k$ contains two pseudotensors on the right-hand side and hence they are compatible with the polar vector on the left.

Another aspect of the use of vectors in Stokes flow is the use made of the dyadic product (also called the outer product). It is convenient to take a quantity such as $(\boldsymbol{U} \cdot \boldsymbol{x})\boldsymbol{x}$ and write it $\boldsymbol{U} \cdot (\boldsymbol{x}\boldsymbol{x})$ with the understanding that the evaluation has not changed. The first way of writing leads one to think in terms of the scalar $\boldsymbol{U} \cdot \boldsymbol{x}$ being constructed first, and then the result used as the coefficient of the vector $\boldsymbol{x}$; the second way leads one to think of a dot product being taken between a vector $\boldsymbol{U}$ and a second-rank tensor $\boldsymbol{x}\boldsymbol{x}$. For a single term, the change looks merely like a complication, but when several terms are being added, the rearrangement allows expressions to be factored that could not be factored otherwise. Thus the expression $(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}+(\boldsymbol{a}\cdot\boldsymbol{d})\boldsymbol{e}$ can be factored by writing

$$(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} + (\boldsymbol{a} \cdot \boldsymbol{d})\boldsymbol{e} = \boldsymbol{a} \cdot (\boldsymbol{b}\boldsymbol{c}) + \boldsymbol{a} \cdot (\boldsymbol{d}\boldsymbol{e}) = \boldsymbol{a} \cdot (\boldsymbol{b}\boldsymbol{c} + \boldsymbol{d}\boldsymbol{e}) \ . \tag{37}$$

Notice that the order of the terms is important. Thus $\boldsymbol{a}\cdot(\boldsymbol{bc}) = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$, but $\boldsymbol{a}\cdot(\boldsymbol{cb}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}$. Also $\boldsymbol{a}\cdot(\boldsymbol{bc}) \neq (\boldsymbol{bc})\cdot\boldsymbol{a}$. Only in special cases such as $\boldsymbol{a}\cdot(\boldsymbol{bb}) = (\boldsymbol{bb})\cdot\boldsymbol{a}$ can the ordering be altered. If the ordering is strictly adhered to, the parentheses can be dropped. If the equations obtained this way become difficult to interpret, remember that they can always be multiplied out and returned to a vector equation.

Dyadics are special sorts of tensors, being tensors built up from vectors. Another special tensor that is very useful is the unit tensor I, defined by $\mathsf{I}\cdot\boldsymbol{a} = \boldsymbol{a}\cdot\mathsf{I} = \boldsymbol{a}$ for any $\boldsymbol{a}$. In subscript notation, I is the Kronecker delta.

One application of these ideas that will be used in the next section is to the breakup of a vector into components parallel and perpendicular to a given direction. Let $\boldsymbol{d}$ be a direction (a unit vector) and let $\boldsymbol{a}$ be any vector. The component of $\boldsymbol{a}$ parallel to $\boldsymbol{d}$ is $(\boldsymbol{a}\cdot\boldsymbol{d})\boldsymbol{d} = (\boldsymbol{dd})\cdot\boldsymbol{a}$. Obviously the perpendicular component is $\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{d})\boldsymbol{d}$. In dyadic notation, one can say that the component of $\boldsymbol{a}$ parallel to $\boldsymbol{d}$ is obtained by taking the dot product of $\boldsymbol{a}$ and $\boldsymbol{dd}$. Similarly, the perpendicular component is obtained by taking the dot product of $\boldsymbol{a}$ with $\mathsf{I}-\boldsymbol{dd}$, since $(\mathsf{I}-\boldsymbol{dd})\cdot\boldsymbol{a} = \boldsymbol{a}-(\boldsymbol{a}\cdot\boldsymbol{d})\boldsymbol{d}$. Notice that the perpendicular property can be proved in general without touching the vector $\boldsymbol{a}$ as follows. Since $\boldsymbol{d}\cdot\boldsymbol{d} = 1$,

$$\boldsymbol{d}\cdot(\mathsf{I}-\boldsymbol{dd})\cdot\boldsymbol{a} = (\boldsymbol{d}\cdot\mathsf{I}-\boldsymbol{d}\cdot\boldsymbol{dd})\cdot\boldsymbol{a} = (\boldsymbol{d}-\boldsymbol{d})\cdot\boldsymbol{a} = 0\ .$$

6 Linearity and tensors

Vector notation greatly simplifies the equations of fluid mechanics, and this simplification brings with it greater understanding and ease of manipulation. Just explore the old books on fluid mechanics, with their equations of motion written out in component form! The linearity of the Stokes equations allows these notational simplifications to go a step further, and mirror in notation more aspects of the flow. Consider the velocity field $\boldsymbol{u}$ around a unit sphere that is moving with velocity $\boldsymbol{U}$. The field is usually written

$$\boldsymbol{u} = \boldsymbol{U}\left(\frac{3}{4r}+\frac{1}{4r^3}\right) + \boldsymbol{x}\frac{\boldsymbol{x}\cdot\boldsymbol{U}}{r^2}\left(\frac{3}{4r}-\frac{3}{4r^3}\right)\ . \tag{38}$$

Here $\boldsymbol{x}$ is a point in the fluid and $r = |\boldsymbol{x}|$. It is fairly easy to see that if the magnitude of the velocity $\boldsymbol{U}$ is doubled, the magnitude of $\boldsymbol{u}$ at

every point is doubled also. More can be seen, however, if the equation is rewritten using the results of the previous section.

$$\boldsymbol{u} = \left[\mathsf{I} \left(\frac{3}{4r} + \frac{1}{4r^3} \right) + \frac{\boldsymbol{x}\boldsymbol{x}}{r^2} \left(\frac{3}{4r} - \frac{3}{4r^3} \right) \right] \cdot \boldsymbol{U} \ . \tag{39}$$

The new feature of this equation is the fact that the brackets do not contain $\boldsymbol{U}$; the effects of the velocity of the sphere and the geometry of the sphere have been separated in the presentation of the equation.

To put it another way, in one dimension a scalar quantity α depends linearly on a scalar quantity β when $\alpha = k\beta$, and moreover the coefficient of proportionality k is also scalar. In (39) there is a similar linear relationship $\boldsymbol{u} = \mathsf{K} \cdot \boldsymbol{U}$, only now the linking coefficient has become a tensor, because $\boldsymbol{u}$ and $\boldsymbol{U}$ are not in the same direction (as a scalar coefficient would imply).

A second example is provided by the drag on a non-spherical particle. The force $\boldsymbol{F}$ is linearly related to the velocity of the particle $\boldsymbol{U}$, but not necessarily in the same direction, as is shown by writing $\boldsymbol{F} = \mathsf{A} \cdot \boldsymbol{U}$. The tensor A is called a resistance tensor and it contains only geometrical factors and information about the particle shape, but nothing about the velocity. This is some help, but A still contains 6 scalar components (not 9, because the reciprocal theorem shows it is symmetric), and could be a difficult object to work with. If, however, the particle is axisymmetric, A can be simplified further. Suppose the axis of symmetry is $\boldsymbol{d}$ (a unit vector), then

$$\mathsf{A} = K_1 \boldsymbol{d}\boldsymbol{d} + K_2 \mathsf{I} \ , \tag{40}$$

where K_1 and K_2 are scalars based on the shape. The proof first extracts the component of velocity parallel to $\boldsymbol{d}$. From above, this is $\boldsymbol{U} \cdot \boldsymbol{d}\boldsymbol{d}$. By symmetry, the force is parallel to this component, so $\boldsymbol{F} = K_{\parallel} \boldsymbol{U} \cdot \boldsymbol{d}\boldsymbol{d}$. The velocity perpendicular to $\boldsymbol{d}$ is $\boldsymbol{U} \cdot (\mathsf{I} - \boldsymbol{d}\boldsymbol{d})$ and in this case too, the force is parallel to the velocity component. Therefore $\boldsymbol{F} = K_{\perp} \boldsymbol{U} \cdot (\mathsf{I} - \boldsymbol{d}\boldsymbol{d})$. Add these together to get

$$\boldsymbol{F} = K_{\parallel} \boldsymbol{U} \cdot (\boldsymbol{d}\boldsymbol{d}) + K_{\perp} \boldsymbol{U} \cdot (\mathsf{I} - \boldsymbol{d}\boldsymbol{d}) = \left[K_{\parallel} \boldsymbol{d}\boldsymbol{d} + K_{\perp} (\mathsf{I} - \boldsymbol{d}\boldsymbol{d}) \right] \cdot \boldsymbol{U} \ , \tag{41}$$

and thus $K_2 = K_{\perp}$ and $K_1 = K_{\parallel} - K_{\perp}$.

The general method can be illustrated with a more general example. Consider an axisymmetric particle that is translating with velocity $\boldsymbol{V}$ and rotating with angular velocity $\boldsymbol{\Omega}$. Since the boundary conditions are

linear in these quantities (cf. equation (2) above), the force is a linear combination of them

$$\boldsymbol{F} = \mathbf{A} \cdot \boldsymbol{U} + \mathbf{B} \cdot \boldsymbol{\Omega} \ . \tag{42}$$

Since $\boldsymbol{F}$, $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are vectors, the coefficients are second-rank tensors, as shown. Furthermore, since $\boldsymbol{\Omega}$ is an axial vector, $\mathbf{B}$ will have to be a pseudotensor. The simplification of $\mathbf{A}$ and $\mathbf{B}$ can be achieved in two ways. Happel & Brenner (1963) turned to the fundamental definition of vectors and tensors given above, and studied the behaviour of the components of the tensors under different transformations. By choosing transformations based on the symmetries of the geometries, they deduced the forms of the tensors $\mathbf{A}$ and $\mathbf{B}$. Here, a different approach is used, one that can be understood by making an analogy with dimensional analysis.

Dimensional analysis is based on the idea that an equation must be dimensionally consistent: each term must have the same dimensions. Furthermore, only quantities relevant to the problem can appear. In a similar way, vector equations must have consistent vectorial parts. The first step, then, is to list the vectors upon which $\mathbf{A}$ and $\mathbf{B}$ can depend. Since the velocity and rotation have already been taken care of in the linear expression, only the axis of symmetry $\boldsymbol{d}$ is left. Therefore, $\mathbf{A}$ and $\mathbf{B}$ must be built from $\boldsymbol{d}$ and the two tensors $\mathbf{I}$ and $\boldsymbol{\epsilon}$. Since $\mathbf{A}$ links $\boldsymbol{F}$ and $\boldsymbol{V}$, both polar vectors, it cannot contain $\boldsymbol{\epsilon}$. Since it is second rank, it will have to contain $\boldsymbol{d}$ in the form $\boldsymbol{dd}$ and contain $\mathbf{I}$. The only combination possible was given in equation (40). The scalar coefficients will contain the remainder of the geometric information.

Now $\mathbf{B}$ joins a polar vector and an axial vector and therefore must contain $\boldsymbol{\epsilon}$. The only way to get a second-rank tensor is to contract the third-rank $\boldsymbol{\epsilon}$ with the vector $\boldsymbol{d}$. Thus the desired form is $B_{ij} = K_3 \epsilon_{ijk} d_k$, and again the scalar coefficient contains the remainder of the geometric information.

As a final example, consider an axisymmetric particle in a uniform rate-of-strain field, that is, a shear field without vorticity. The flow is given by $\boldsymbol{u} = \mathbf{E} \cdot \boldsymbol{x}$, where $\mathbf{E}$ is a symmetric and traceless second-rank tensor. For the force $\boldsymbol{F}$ to depend linearly on $\mathbf{E}$, a third-rank tensor must be introduced. In subscript notation,

$$F_i = G_{ijk} E_{jk} \ .$$

Now G_{ijk} must be built from d_i and δ_{ij}, since it joins polar vectors. The

most general form for G_{ijk} is then

$$G_{ijk} = K_4 d_i d_j d_k + K_5 d_i \delta_{jk} + K_6 d_j \delta_{ik} + K_7 d_k \delta_{ij} \ . \qquad (43)$$

Since E_{jk} is traceless, the term $K_5 d_i \delta_{jk}$ cannot contribute to F_i and may therefore be dropped. Similarly, because E_{jk} is symmetric, the terms $K_6 d_j \delta_{ik}$ and $K_7 d_k \delta_{ij}$ will always contribute to F_i in the form $K_6 + K_7$, so one term could be dropped. However, for reasons of asthetics, they are more commonly set equal. Thus G_{ijk} is reduced to depending on two independent scalar functions. As in the case of **A**, there is some arbitrariness in assigning the two functions, cf. equations (40) and (41), and different authors might vary in their choices.

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