This article was published in The Mathematical Scientist, **21**, 1–7 (1996). The pagination here is the same as the published version.

Unwinding the branches of the Lambert W function

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Abstract

An algebraic relation is derived that allows the different branches of the Lambert W function to be concisely distinguished. The derivation relies on the unwinding number, which is defined here, for the manipulation of elementary functions in the complex plane.

Key words: Lambert W function; unwinding number.
AMS 1991 Subject Classification: Primary 33E99

1. Introduction

Interest in the Lambert W function goes back at least to Lambert and Euler, albeit that these two mathematicians studied it only indirectly, but the function did not receive a permanent name until it was included in the library of Maple, the computer algebra system. After gaining its name, it gained recognition. Many of the applications of W were found because of users discovering that Maple had used W in the solution of a problem they had posed. In addition to its applicability, the function has rich mathematical properties. It is defined to be the solution $W(z)$ of

$$W(z)e^{W(z)} = z \ , \tag{1}$$

where z is a complex variable. The history, applications and properties of W were recently reviewed in Corless *et al.*[3], and so just two examples of the uses of W are given here, in order to convey the flavour of its applications.

A common way to meet W is through the problem of iterated exponentiation, which is the evaluation of

$$h(z) = z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} \ ,$$

whenever it makes sense. The function $h(z)$ can be found in closed form by solving the

Received 29 Nov 1995; revised 8 Jan 1996
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equation $h(z) = z^{h(z)}$ for $h(z)$ (using Maple, for example), and getting

$$h(z) = -\frac{W(-\log z)}{\log z} . \tag{2}$$

For more details and references on the problem, see Baker & Rippon [2].

Another application is the solution of linear constant-coefficient delay equations [8]. Consider the simple delay equation

$$\dot{y}(t) = ay(t-1) , \tag{3}$$

subject to the condition on $0 \le t \le 1$ that $y(t) = f(t)$, a known function. Direct substitution shows that $\exp(W(a)t)$ is a solution of (3). The starting condition can be matched by noting that (1) does not define W uniquely. If complex values are considered, multiple solutions exist, denoted $W_k(a)$, for k an integer. Then, by linearity, a solution of (3) is

$$y = \sum_{k=-\infty}^{\infty} c_k \exp(W_k(a)t) , \tag{4}$$

and the c_k can be determined to match $f(t)$. One sees immediately that the solution will grow exponentially if any of the $W_k(a)$ have positive real part, and this leads to important stability theorems in the theory of delay equations.

As a prelude to describing the branches W_k, it is useful to consider the branches of some elementary functions that are multivalued. The manipulation of multivalued functions in the complex plane has been the subject of renewed attention in recent years, in an attempt to impose some uniformity on the operations of hand-held calculators and computer programs. As a result, there is now substantial agreement on the principal branches of the elementary functions [7]. The principal argument of a complex number z satisfies $-\pi < \arg z \le \pi$, and the principal branch of its (natural) logarithm is defined to be $\ln z = \ln|z| + i \arg z$. The kth branch of the logarithm is written $\ln_k z = \ln z + 2\pi i k$, implying that $\ln_0 z$ is another representation of the principal branch. (On the matter of notation, notice that although $\log_k$ means logarithm to the base k, there is no standard interpretation of $\ln_k$, so there need be no confusion of meaning.) The notation $\mathrm{Ln}\, z$ denotes an unspecified branch. The range of each $\ln_k z$ in the complex plane is shown in figure 1. Plotted horizontally is the real part of the logarithm: $\Re \ln_k z = \ln|z|$. The imaginary part, $\Im \ln_k z = \arg z + 2\pi k$, is plotted vertically. The points that form the boundary between two branches belong to the region below them, because of the definition of $\arg z$. This closure also agrees with the 'counter-clockwise continuous' convention (CCC) of Kahan [7].

Figure 2 shows the complex range of each W_k. The horizontal axis is $\xi = \Re W$, the real part of the appropriate branch of W, and the vertical axis is $\eta = \Im W$. The branch boundaries obey either $\eta = 0$ or $\xi = -\eta \cot \eta$ and correspond to cuts in the z plane along portions of the negative real axis. The points on the boundaries belong to the branch below them, which closure rule again satisfies the counter-clockwise continuous (CCC) convention [7]. The negative real axis is divided at $\xi = -1$, with $\xi \ge -1$ belonging to branch 0 and $\xi < -1$ belonging to branch $k = -1$. It should be noticed that the dashed asymptotes at odd multiples of π coincide with the branch boundaries in figure 1. This is a consequence of the fact that for $|z| \to \infty$, $W_k(z) \to \ln_k z$.

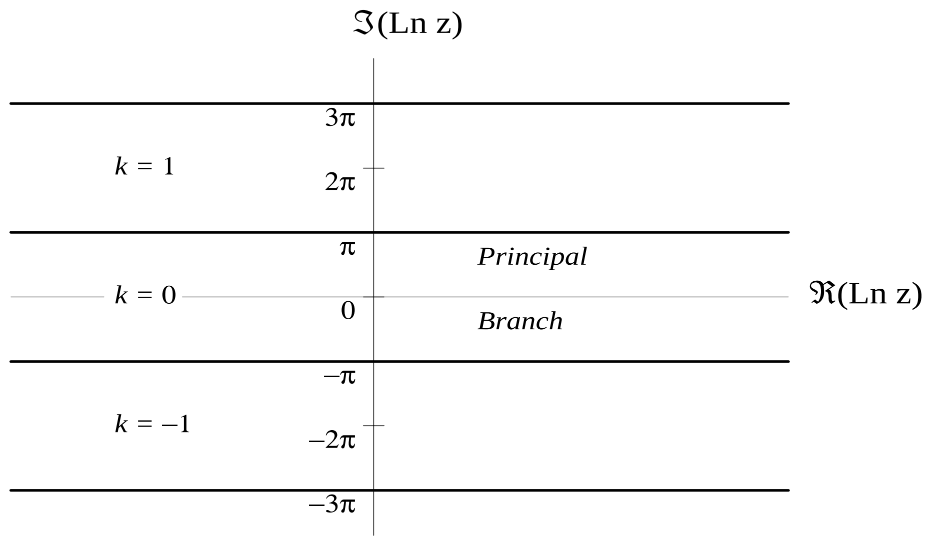

Figure 1. The ranges of the branches of the multivalued logarithm $\operatorname{Ln} z$. The heavy lines are the boundaries between the branches, and the value of k for each branch is shown. Branch k corresponds to $\ln_k z = \ln z + 2\pi i k$, where $\ln z = \ln_0 z$ is the principal branch of logarithm.

The current definition of the branches is a little long-winded, and it would be a great convenience if it could be replaced, or at least supplemented, by an algebraic relation that directly discriminates between the branches of W. Section 3 derives such a relation. It is more concise, as a definition, although the original definition is more graphical. To derive the result, we have used a method currently being explored in connection with computer algebra systems.

Computer algebra systems, and humans for that matter, face the problem of how to handle the fact that in the complex plane $\ln(zw) \neq \ln z + \ln w$, when logarithms are interpreted as principal branch. Any scheme should be suitable for automatic operation. One proposed solution is to transfer the problem to the design of the user interface to the system, and not develop any new mathematics. This possibility has been discussed, amongst other issues, in Corless & Jeffrey [4]. Although plausible in the computer-algebra context, an interface solution does nothing to alleviate the mathematical difficulty. Here, a mathematical idea called the unwinding number is described and applied. Similar ideas have been discussed informally, particularly in electronic forums such as newsnet, but have not appeared in print, although an unnamed function similar to the unwinding number made a cameo appearance in the book by Apostol [1, theorem 1.54].

2. Definition and properties of the unwinding number

The unwinding number $\mathcal{K}(z)$ is an integer function defined using the principal-value logarithm. The definition is

$$\ln(e^z) = z + 2\pi i \mathcal{K}(z) \ . \tag{5}$$

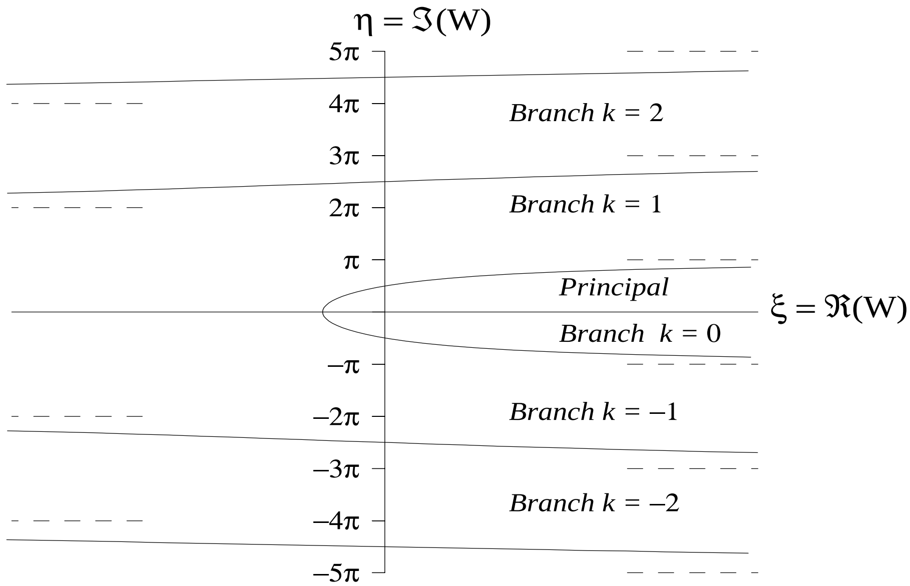

Figure 2. The ranges of the branches of W. Each branch is given a number, the principal branch being numbered 0. The boundaries of the branches are asymptotic to the dashed lines, which are horizontal at integer multiples of π. The boundary curves obey either $\xi = -\eta \cot \eta$, for $2n\pi < |\eta| < (2n+1)\pi$, or $\eta = 0$ for $\xi \le -1$.

Equivalent definitions use the floor function or the round function. If $\Im z$ is the imaginary part of z, then

$$\mathcal{K}(z) = \left\lfloor \frac{\pi - \Im z}{2\pi} \right\rfloor = -\mathcal{R}\left(\frac{\Im z}{2\pi}\right) . \tag{6}$$

Here, $\mathcal{R}$ is the round function with half-integers rounded down. Particular values of $\mathcal{K}$ are

$$\mathcal{K}(x+iy) = \begin{cases} 1, & \text{when } -3\pi < y \le -\pi \ , \\ 0, & \text{when } -\pi < y \le \pi \ , \\ -1, & \text{when } \pi < y \le 3\pi \ , \\ -n, & \text{when } (2n-1)\pi < y \le (2n+1)\pi \ . \end{cases}$$

Only those properties of $\mathcal{K}$ that are of immediate relevance will be explored here. From (6), it is clear that $\mathcal{K}(z) = \mathcal{K}(i\Im z)$, and also

$$\mathcal{K}(z + 2\pi i n) = \mathcal{K}(z) - n \ , \quad \text{for } n \text{ an integer.} \tag{7}$$

This relation is very useful for removing one unwinding number from the argument of another. The application of $\mathcal{K}$ in this paper is to the manipulation of logarithms, as summarized by the following formulae.

$$\ln(zw) = \ln z + \ln w + 2\pi i \mathcal{K}(\ln z + \ln w) \ . \tag{8}$$

$$\ln(z/w) = \ln z - \ln w + 2\pi i \mathcal{K}(\ln z - \ln w) \ . \tag{9}$$

Both of these apply for any complex numbers z and w. The derivation of (8) is effected by taking logs of $zw = \exp(\ln z + \ln w)$. The special case $w = 1$ implies

$$\mathcal{K}(\ln z) = 0 \ . \tag{10}$$

While giving an example of working with $\mathcal{K}$, we can derive a result that will be used in section 4. If a and b are complex numbers for which $|\arg b - \arg a| < \pi$, with arg the principal argument, then

$$\ln b = \ln a + \ln \frac{b}{a} \ . \tag{11}$$

The derivation uses (8), (9) and (7).

$$\begin{aligned} \ln b &= \ln a + \ln(b/a) + 2\pi i \mathcal{K}\left(\ln a + \ln(b/a)\right) \\ &= \ln a + \ln(b/a) + 2\pi i \mathcal{K}\left(\ln a + \ln b - \ln a + 2\pi i \mathcal{K}(\ln b - \ln a)\right) \\ &= \ln a + \ln(b/a) + 2\pi i \mathcal{K}(\ln b) - 2\pi i \mathcal{K}(\ln b - \ln a) \ . \end{aligned}$$

The first unwinding number is zero by (10) and the second by the condition on a and b.

3. A new relation for Lambert W

The main result of this article is, in the notation introduced,

$$W_k(z) + \ln W_k(z) = \begin{cases} \ln z \ , & \text{for } k = -1 \text{ and } z \in [-1/e, 0) \ , \\ \ln_k z & \text{otherwise.} \end{cases} \tag{12}$$

The proof takes logarithms of (1). Omitting the argument of $W_k(z)$ for clarity, we have

$$\begin{aligned} \ln z &= \ln\left(W_k \exp W_k\right) = \ln W_k + \ln\exp W_k + 2\pi i \mathcal{K}(\ln W_k + \ln\exp W_k) \\ &= \ln W_k + W_k + 2\pi i \mathcal{K}(W_k) + 2\pi i \mathcal{K}(\ln W_k + W_k + 2\pi i \mathcal{K}(W_k)) \ . \end{aligned}$$

Equation (7) now cancels two of the unwinding numbers.

$$\ln z = \ln W_k + W_k + 2\pi i \mathcal{K}(\ln W_k + W_k) \ .$$

So it is required to show $\mathcal{K}(\ln W_k + W_k) = -k$ for $k \neq -1$, and this requires showing $(2k-1)\pi < \arg W_k + \Im W_k \leq (2k+1)\pi$. Put $z = re^{i\theta}$, with $-\pi < \theta \leq \pi$, and $W_k = \xi + i\eta$ in (1) and separate real and imaginary parts.

$$r\cos\theta = e^{\xi}\left(\xi\cos\eta - \eta\sin\eta\right) \ , \tag{13}$$

$$r\sin\theta = e^{\xi}\left(\eta\cos\eta + \xi\sin\eta\right) \ . \tag{14}$$

For $\theta \neq 0$ or π, divide (13) by (14).

$$\cot\theta = \frac{\cot\arg W_k \cot\eta - 1}{\cot\arg W_k + \cot\eta} = \cot(\arg W_k + \Im W_k) \ .$$

Thus θ and $\arg W_k + \Im W_k$ differ by an integer multiple of π. Furthermore, by continuity, this multiple is the same for all r and θ, except possibly when W is real and negative, because then $\arg W$ is discontinuous. Putting the exceptions to one side, we identify the multiple of π by fixing θ and considering $r \to \infty$. Since W is asymptotically similar to logarithm as $|z| \to \infty$, we have, for θ fixed,

$$\lim_{r\to\infty} (\arg W_k + \Im W_k) = \lim_{r\to\infty} \eta = \theta + 2k\pi \ ,$$

since $\arg W_k \to 0$ as $|z| \to \infty$. Therefore for $\theta \neq 0, \pi$ and for all r,

$$\arg W_k + \Im W_k = \theta + 2\pi k \ . \tag{15}$$

The special case $\theta = 0$ requires $k = 0$ and is trivial; the special case $\theta = \pi$ requires either $k = -1$ or $k = 0$. If $k = -1$, then $\theta = \pi$ corresponds to $W_{-1} \leq -1$ when $r \leq 1/e$, and then $\arg W_{-1}$ is discontinuous at $\theta = \pi$ [3, figures 7 and 8]. Hence

$$\arg W_{-1}(z) + \Im W_{-1}(z) = \pi \qquad \text{for} - 1/e \leq z < 0 \ ,$$

and $\mathcal{K} = 0$, as stated in the theorem. For the $k = 0$ branch, $W_0 \geq -1$ also corresponds to $\theta = \pi$ and $r \leq 1/e$, but in this case, (15) continues to apply.

4. Application

De Bruijn [5] obtained an asymptotic expansion for the real values of $W_0(x)$, for x real, when $x \to \infty$. This was generalized to all branches of $W(z)$ for complex z in the case $|z| \to \infty$ by Corless *et al.* [3]. In both cases, the derivations were based on a quantity v, for which an equation was derived that was valid in the asymptotic region. Here, (12) and (11) are used to obtain an equation for v that is valid for all z and all branches.

Generalizing de Bruijn, we define v_k by

$$W_k(z) = \ln_k z - \ln\ln_k z + v_k(z) \ . \tag{16}$$

For $k = 0$, v_0 is unbounded at $z = 0, 1$. We show that (suppressing the argument of v_k)

$$v_k + \ln\left(1 - \frac{\ln\ln_k z}{\ln_k z} + \frac{v_k}{\ln_k z}\right) = \begin{cases} 2\pi i & \text{for } k = 0 \text{ and } z \in (0,1), \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The proof treats separately the three cases $\arg W_k = 0, \pi$ and $\arg W_k$ not equal to these values. If $\arg W \neq 0, \pi$, then substituting (16) into (12) gives

$$\ln_k z - \ln\ln_k z + v_k + \ln(\ln_k z - \ln\ln_k z + v_k) = \ln_k z \ .$$

For all branches, $\arg W_k$ and $\arg \ln_k$ have the same sign, and so (11) applies, and results in (17).

The special case $\arg W_k = 0$ implies $k = 0$ and z is real and positive, and checking this case is straightforward. Finally, if $\arg W = \pi$, then k can be 0 or -1, and, for either value of k, $-1/e \leq z < 0$. Since $k = -1$ is a special case in (12), one might expect to see a separate line for it in (17), but a factor $2\pi i$ is introduced because of the two conditions $\arg W_{-1} = \pi$ but $\arg L_{-1} < 0$, and this factor cancels the same factor in (12).

Exponentiating (17) casts it in a more familiar form due to de Bruijn.

$$e^{-v} - 1 + \frac{\ln \ln_k z}{\ln_k z} - \frac{v}{\ln_k z} = 0 \ . \tag{18}$$

From (17), $-\pi \le \Im v \le \pi$, and any solution of (18) must satisfy this constraint to be relevant. The last equation has been the starting point for several series expressions, both asymptotic and convergent, for the Lambert W function [6].

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