Lecture 02 (September 28, 2009)

4 Ordinary chain complexes: homotopy theory

Definition 4.1. Say that a map $f : C \to D$ in $Ch_+(R)$ is a

- *weak equivalence* if f is a homology isomorphism,
- *fibration* if $f : C_n \to D_n$ is surjective for $n > 0$,
- *cofibration* if f has the left lifting property (LLP) with respect to all morphisms of $Ch_+(R)$ which are simultaneously fibrations and weak equivalences.

In different words, a map $i : A \to B$ of chain complexes is a cofibration if given any solid arrow commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & D \end{array}$$

with $p : C \to D$ a fibration and a weak equivalence, the dotted arrow exists making the diagram commute.

Remark 4.2. Morphisms which are simultaneously fibrations and weak equivalences are called *trivial fibrations*. Similarly, morphisms which are simultaneously cofibrations and weak equivalences are *trivial cofibrations*. This terminology appears throughout homotopy theory.

All trivial fibrations p have the *right lifting property* with respect to all cofibrations i.

Here are some special chain complexes and chain maps:

- $R(n)$ is the chain complex consisting of a copy of the free R-module R, concentrated in degree n:

  $$\cdots \to 0 \to 0 \to \overset{n}{R} \to 0 \to 0 \to \dots$$

  There is a natural R-module isomorphism

  $$\hom_{Ch_+(R)}(R(n), C) \cong Z_n(C).$$

- $R\langle n+1\rangle$ is the chain complex

  $$\cdots \to 0 \to \overset{n+1}{R} \xrightarrow{1} \overset{n}{R} \to 0 \to \dots$$

- There is a natural R-modules isomorphism

  $$\hom_{Ch_+(R)}(R\langle n+1\rangle, C) \cong C_{n+1}.$$

- There is a morphism $\alpha : R(n) \to R\langle n+1 \rangle$ given by the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow[1]{} & R & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

  Equivalently, α classifies the cycle

$$1 \in R\langle n+1 \rangle_n.$$

Lemma 4.3. *Suppose that $p : A \to B$ is a fibration and that $i : K \to A$ is the inclusion of the kernel of p. Then there is a long exact sequence*

$$\ldots \xrightarrow{p_*} H_{n+1}(B) \xrightarrow{\partial} H_n(K) \xrightarrow{i_*} H_n(A) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} \ldots$$
$$\ldots \xrightarrow{\partial} H_0(K) \xrightarrow{i_*} H_0(A) \xrightarrow{p_*} H_0(B).$$

Proof. Suppose that $j : \mathrm{im}(p) \subset B$ is the inclusion of the image of p in B, and write $\pi : A \to \mathrm{im}(p)$ for the induced epimorphism. Then $H_n(\mathrm{im}(p)) = H_n(B)$ for $n > 0$, and there is a commutative diagram

$$\begin{array}{ccc} H_0(A) & \xrightarrow{\quad p_* \quad} & H_0(B) \\ & {\scriptstyle \pi_*}\searrow \quad \nearrow{\scriptstyle i_*} & \\ & H_0(\mathrm{im}(p)) & \end{array}$$

in which π_* is an epimorphism and i_* is a monomorphism (exercise). Then the desired long exact sequence is constructed from the long exact sequence in homology for the short exact sequence

$$0 \to K \xrightarrow{i} A \xrightarrow{\pi} \mathrm{im}(p) \to 0$$

by composing with the monomorphism

$$i_* : H_0(\mathrm{im}(p)) \to H_0(B)$$

in degree 0. □

Observation: The map $p : A \to B$ is a fibration if and only if p has the right lifting property with respect to all maps $0 \to R\langle n+1 \rangle$, $n \geq 0$.

This means that the dotted arrow exists, making the diagram commute, in all solid arrow diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow p \\ R\langle n+1 \rangle & \longrightarrow & B \end{array}$$

Consequence: The map $0 \to R\langle n+1 \rangle$ is a trivial cofibration for all $n \geq 0$.

In effect, this map has the left lifting property with respect to all fibrations, hence with respect to all trivial fibrations.

Lemma 4.4. *The map* $0 \to R(n)$ *is a cofibration.*

Proof. We want to show that every trivial fibration $p : A \to B$ induces an epimorphism $Z_n(A) \to Z_n(B)$ for all $n \geq 0$. If $x \in B_n$ is a cycle, then there is a cycle $z \in A_n$ and a chain $w \in B_{n+1}$ such that $p(z) = x + \partial w$. There is a chain $v \in A_{n+1}$ such that $p(v) = w$ since p is surjective in non-zero degrees. Thus $p(z - \partial(v)) = x$. □

Some language: A chain complex A is said to be *cofibrant* if the map $0 \to A$ is a cofibration. Thus, the objects $R\langle n+1\rangle$ and $R(n)$ are cofibrant.
Dually, all chain complexes are *fibrant*, because all chain maps $C \to 0$ are fibrations.

Proposition 4.5. *A map* $p : A \to B$ *is a fibration and a weak equivalence if and only if* $p : A_0 \to B_0$ *is a surjection and* p *has the right lifting property with respect to all maps* $\alpha : R(n) \to R\langle n+1\rangle$.

Corollary 4.6. *The map* $\alpha : R(n) \to R\langle n+1\rangle$ *is a cofibration.*

Proof of Proposition 4.5. Suppose that $p : A \to B$ is a trivial fibration.

Chase the comparison of exact sequences

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\partial} & A_0 & \longrightarrow & H_0(A) & \longrightarrow & 0 \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \cong} & & \\
B_1 & \xrightarrow[\partial]{} & B_0 & \longrightarrow & H_0(B) & \longrightarrow & 0
\end{array}$$

keeping in mind that $p : A_1 \to B_1$ is surjective to show that $p : A_0 \to B_0$ is surjective.

Suppose given a commutative diagram

$$\begin{array}{ccc}
R(n) & \xrightarrow{x} & A \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle p} \\
R\langle n+1\rangle & \xrightarrow[y]{} & B
\end{array}$$

Choose $z \in A_{n+1}$ such that $p(z) = y$. Then $x - \partial(z)$ is a cycle of K, and K is acyclic by a long exact sequence argument so there is a $v \in K_{n+1}$ such that $\partial(v) = x - \partial(z)$. But then $\partial(z+v) = x$ and $p(v+z) = p(v) = y$, so the chain $v+z$ is the desired lift.

Suppose that $p : A_0 \to B_0$ is surjective and that p has the right lifting property with respect to all $R(n) \to R\langle n+1\rangle$.

The solutions of the lifting problems

$$\begin{array}{ccc}
R(n) & \xrightarrow{0} & A \\
\downarrow & \nearrow & \downarrow{\scriptstyle p} \\
R\langle n+1\rangle & \xrightarrow[x]{} & B
\end{array}$$

show that p is surjective on all cycles, while the solutions of the lifting problems

$$\begin{array}{ccc} R(n) & \xrightarrow{x} & A \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ R\langle n+1\rangle & \xrightarrow[y]{} & B \end{array}$$

show that p induces a monomorphism in all homology groups. It follows that p is a weak equivalence.

Now look at the diagram

$$\begin{array}{ccccc} Z_{n+1}(A) & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & Z_n(A) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Z_{n+1}(B) & \longrightarrow & B_{n+1} & \xrightarrow[\partial]{} & Z_n(B) \end{array}$$

and take $x \in B_{n+1}$. Then $\partial(x) = p(v)$ for some $v \in Z_n(A)$ since p is surjective on cycles, and $[\partial(x)] = 0$ in $H_n(B)$ implies that $[v] = 0 \in H_n(A)$, so that $v = \partial(w)$ for some $w \in A_{n+1}$. But then $\partial(x - p(w)) = 0$, so there is $z \in Z_{n+1}(A)$ such that $p(z) = x - p(w)$, and so $x = p(z - w)$. In particular, p is surjective in all degrees and is therefore a fibration. $\square$

Proposition 4.7. *Every chain map $f : C \to D$*

has two factorizations

$$\begin{array}{ccccc} & & E & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle p} & \\ C & & \xrightarrow{\quad f \quad} & & D \\ & {\scriptstyle j}\searrow & & \nearrow{\scriptstyle q} & \\ & & F & & \end{array}$$

where

1) *p is a fibration and i is a monomorphism, a weak equivalence and has the left lifting property with respect to all fibrations, and*
2) *q is a trivial fibration and j is a monomorphism and a cofibration.*

Proof. For 1) form the factorization

$$\begin{array}{ccccc} & & C \oplus (\oplus_{x \in D_{n+1}, n \geq 0} R\langle n+1 \rangle) & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle p} & \\ C & & \xrightarrow[\quad f \quad]{} & & D \end{array}$$

The map p is the sum of the map f and all classifying maps for chains x in all non-zero degrees. It is therefore surjective in non-zero degrees and is thus a fibration. The map i is the inclusion of a direct summand with acyclic cokernel, and is therefore a monomorphism and a weak equivalence. It is also a direct sum of maps which have the left lifting property with respect to all fibrations, and therefore has that same lifting property.

For 2), recall that a map $q : A \to B$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations $R(n) \to R\langle n+1\rangle$, $n \geq -1$ (where $R(-1) \to R\langle 0\rangle$ is notation for the map $0 \to R(0)$).

Consider the set of all diagrams

$$D: \qquad \begin{array}{ccc} R(n_D) & \xrightarrow{\alpha_D} & C \\ \downarrow & & \downarrow{\scriptstyle f=q_0} \\ R\langle n_D+1\rangle & \xrightarrow[\beta_D]{} & D \end{array}$$

and form the pushout

$$\begin{array}{ccc} \oplus_D R(n_D) & \xrightarrow{(\alpha_D)} & C_0 \\ \downarrow & & \downarrow{\scriptstyle j_1} \\ \oplus_D R\langle n_D+1\rangle & \xrightarrow[(\theta_D)]{} & C_1 \end{array}$$

where $C = C_0$. Then j_1 is a monomorphism and cofibration, because the collection of all such maps is closed under direct sum and pushout. Then the maps β_D induce a map $q_1 : C_1 \to D$ which makes the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{j_1} & C_1 \\ & {\scriptstyle q_0}\searrow & \downarrow{\scriptstyle q_1} \\ & & D \end{array}$$

commute.

Note that every lifting problem D as above is solved in C_1, in the sense that the diagram

$$\begin{array}{ccccc} R(n_D) & \xrightarrow{\alpha_D} & C_0 & \xrightarrow{j_1} & C_1 \\ \downarrow & & & \nearrow_{\theta_D} & \downarrow q_1 \\ R\langle n_D + 1\rangle & & \xrightarrow{\beta_D} & & D \end{array}$$

commutes.

Repeat this process inductively for the maps q_i to produce a string of factorizations

$$\begin{array}{ccccccc} C_0 & \xrightarrow{j_1} & C_1 & \xrightarrow{j_2} & C_2 & \xrightarrow{j_3} & \cdots \\ {\scriptstyle q_0}\downarrow & \swarrow_{q_1} & & \swarrow_{q_2} & & & \\ D & & & & & & \end{array}$$

Let $F = \varinjlim_i C_i$, so that f has an induced factorization

$$\begin{array}{ccc} C & \xrightarrow{j} & F \\ & {\scriptstyle f}\searrow & \downarrow q \\ & & D \end{array}$$

Then the map j is a cofibration and a monomorphism, because all maps j_k have these properties and the collection of all such maps is closed under infinite composition.

Finally, given a diagram

$$\begin{array}{ccc} R(n) & \xrightarrow{\alpha} & F \\ \downarrow & & \downarrow q \\ R\langle n+1\rangle & \xrightarrow[\beta]{} & D \end{array}$$

The map α factors through some stage of the filtered colimit defining F, so that α is a composite

$$R(n) \xrightarrow{\alpha'} C_k \to F$$

for some k. But then the lifting problem

$$\begin{array}{ccc} R(n) & \xrightarrow{\alpha'} & C_k \\ \downarrow & & \downarrow q_k \\ R\langle n+1\rangle & \xrightarrow[\beta]{} & D \end{array}$$

is solved in C_{k+1}, and hence in F. □

Remark 4.8. This last proof is a "small object argument". Basically, the idea is that the objects $R(n)$ are small in the sense that $\hom(R(n), \ )$ commutes with filtered colimits.

Corollary 4.9. *1) Every cofibration is a monomorphism.*

2) Suppose that $j : C \to D$ is a cofibration and a weak equivalence. Then j has the left lifting property with respect to all fibrations.

Proof. 2) The map j has a factorization

$$\begin{array}{ccc} C & \xrightarrow{i} & F \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle p} \\ & & D \end{array}$$

where i has the left lifting property with respect to all fibrations, and p is a fibration. The map p is also a trivial fibration, so the lifting exists in the diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & F \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ D & \xrightarrow[1]{} & D \end{array}$$

since j is a cofibration. It follows that j is a retract of a map (namely i) which has the left lifting property with respect to all fibrations, and so j has the same property.

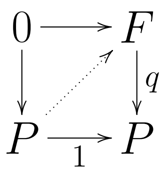
1) is an exercise. □

Suppose that P is an ordinary chain complex. Then Proposition 4.7 says that the map $0 \to P$ has a factorization

$$\begin{array}{ccc} 0 & \xrightarrow{j} & F \\ & \searrow & \downarrow{\scriptstyle q} \\ & & P \end{array}$$

where j is a cofibration (so that F is cofibrant) and q is a trivial fibration, hence a weak equivalence. In the proof of Proposition 4.7 for the corresponding factorization of a chain map $f : C \to D$, C_{k+1} is constructed from C_k degreewise by taking a direct sum with some (large) free R-module. It follows that each R-module F_n in the "resolution" F of P is free, so that F is a free resolution of P.

If the chain complex P happens to be cofibrant, then the lifting exists in the diagram



since $0 \to P$ is a cofibration and q is a trivial fibration. It follows that all chain modules P_n are direct summands of free modules and are therefore projective. This result has a converse, giving the following:

Lemma 4.10. *An ordinary chain complex P is cofibrant if and only if all modules of chains P_n are projective.*

Proof. We have to show that P is cofibrant if all P_n are projective.

Suppose that $p : A \to B$ is a trivial fibration. Then $p : A_n \to B_n$ is surjective for all $n \geq 0$

by Proposition 4.5, and has acyclic kernel by a long exact sequence argument (Lemma 4.3) Let $i : K \to A$ be the kernel of p. Suppose given a diagram

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow p \\ P & \xrightarrow[f]{} & B \end{array}$$

where P is a complex of projectives. We need to find a chain map $\theta : P \to A$ such that $p\theta = f$.

There is a morphism $\theta_0 : P_0 \to A_0$ so that the diagram

$$\begin{array}{ccc} & & A_0 \\ & \overset{\theta_0}{\nearrow} & \downarrow p_0 \\ P_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

commutes, since p_0 is an epimorphism and P_0 is projective.

Suppose given R-module homomorphisms $\theta_i : P_i \to A_i$ for $i \leq n$ such that $p_i\theta_i = f_i$ for $i \leq n$ and $\partial\theta_i = \theta_{i-1}\partial$ for $1 \leq i \leq n$ (in other words, the morphisms θ_i form a chain map up to degree n).

There is a morphism $\theta'_{n+1} : P_{n+1} \to A_{n+1}$ such that $p_{n+1}\theta'_{n+1} = f_{n+1}$. Then

$$p_n(\partial\theta'_{n+1} - \theta_n\partial) = \partial p_{n+1}\theta'_{n+1} - f_n\partial = 0,$$

so there is a morphism $v : P_{n+1} \to K_n$ such that

$$i_n v = \partial\theta'_{n+1} - \theta_n\partial.$$

At the same time,

$$\partial(\partial\theta'_{n+1} - \theta_n\partial) = 0$$

and K is acyclic, so there is a morphism $w : P_{n+1} \to K_{n+1}$ such that

$$i_n\partial w = \partial\theta'_{n+1} - \theta_n\partial.$$

Then

$$\partial(\theta'_{n+1} - i_{n+1}w) = \theta_n\partial$$

and

$$p_{n+1}(\theta'_{n+1} - i_{n+1}w) = p_{n+1}\theta'_{n+1} = f_{n+1}.$$

In other words the lifting $\{\theta_i\}$ up to degree n can be extended to a lifting up to degree $n+1$, where $\theta_{n+1} = \theta'_{n+1} - i_{n+1}w$. □

Remark 4.11.

- Every chain complex C has a cofibrant (or projective) model, meaning a weak equivalence $p : P \to C$ with P cofibrant, on account of Proposition 4.7.
- Suppose that M is an R-module, and form the chain complex $M(0)$. Then a cofibrant model $P \to M(0)$ is a projective resolution of M in the traditional sense by Lemma 4.10.

- Cofibrant models $P \to C$ are also (more commonly) constructed with Cartan-Eilenberg resolutions [1, 5.7].

5 Closed model categories

A *closed model category* is a category $\mathcal{M}$ equipped with three classes of maps, namely weak equivalences, fibrations and cofibrations, such that the following requirements are satisfied:

CM1 The category $\mathcal{M}$ has all finite limits and colimits.

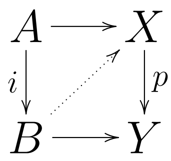
CM2 Given a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & Z & \end{array}$$

of morphisms in $\mathcal{M}$, if any two of f, g and h are weak equivalences , then so is the third.

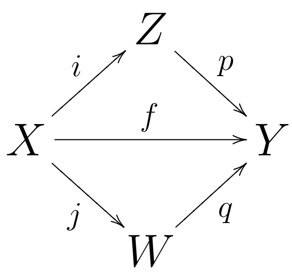
CM3 The classes of cofibrations, fibrations and weak equivalences are closed under retraction.

CM4 Suppose given a commutative solid arrow diagram



such that i is a cofibration and p is a fibration. Then the lifting exists making the diagram commute if either i or p is a weak equivalence.

CM5 Any morphism $f : X \to Y$ of $\mathcal{M}$ has factorizations



where p is a fibration and i is a trivial cofibration, and q is a trivial fibration and j is a cofibration.

Theorem 5.1. *With the definition of weak equivalence, fibration and cofibration given above, $Ch_+(R)$ satisfies the axioms for a closed model category.*

Proof. **CM1**, **CM2** and **CM3** are trivial to verify. **CM5** is Proposition 4.7, and **CM4** is a Corollary 4.9. □

We'll see as time goes by that the general outline of the argument for the closed model structure on the category $Ch_+(R)$ of ordinary chain complexes of R-modules is quite typical.

Exercise: Say that a map $f : C \to D$ of $Ch(R)$ (unbounded chain complexes) is a weak equivalence if it is a homology isomorphism, and is a fibration if all maps $f : C_n \to D_n$, $n \in \mathbb{Z}$ are surjective. A map of unbounded chain complexes is a cofibration if and only if it has the left lifting property with respect to all maps which are fibrations and weak equivalences (aka. trivial fibrations). Show that, with these definitions, $Ch(R)$ has the structure of a closed model category.

References

[1] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.