Lecture 005 (October 30, 2009)

10 Tensor products of chain complexes

Suppose that C is a chain complex of right R-modules and that D is a complex of left R-modules. The *tensor product* $C \otimes_R D$ of these complexes is the chain complex with

$$(C \otimes_R D)_n = \bigoplus_{p+q=n} (C_p \otimes_R D_q),$$

The boundary

$$\partial : (C \otimes_R D)_n \to (C \otimes_R D)_{n-1}$$

is defined on $x \otimes y \in C_p \otimes_R D_q$ (in *bidegree* (p,q)) by

$$\partial(x \otimes y) = (\partial(x) \otimes y) + (-1)^p (x \otimes \partial y).$$

One often sees $|x| = p$ if $x \in C_p$ for the *degree* of x, and so the boundary formula can be written

$$\partial(x \otimes y) = (\partial(x) \otimes y) + (-1)^{|x|} (x \otimes \partial y).$$

It is a simple exercise to show that $\partial^2(x \otimes y)$, so that $C \otimes_R D$ is a chain complex.

Something bigger lurks behind this definition: a *bicomplex* E is an array of abelian groups $E_{p.q}$,

$p, q \geq 0$, together with abelian group homomorphisms

$$\partial_h : E_{p,q} \to E_{p-1,q} \text{ and } \partial_v : E_{p,q} \to E_{p,q-1},$$

such that the following hold:

- $\partial_h^2 = 0$ and $\partial_v^2 = 0$, so that E consists of chain complexes in both the horizontal and vertical directions, and
- $\partial_v \partial_h = \partial_h \partial_v$, so that the *horizontal boundaries* ∂_h and the *vertical boundaries* ∂_h define maps of chain complexes.

The *total complex* $\mathrm{Tot}(E)$ of a bicomplex E is the chain complex with

$$\mathrm{Tot}(E)_n = \bigoplus_{p+q=n} E_{p,q},$$

and with boundary

$$\partial x = \partial_h(x) + (-1)^p \partial_v(x)$$

for $x \in E_{p,q}$.

For $x \in E_{p,q}$, we would say that x has *bidegree* (p, q). I'm also inclined to say that x has *horizontal degree* p and *vertical degree* q, but "horizontal" and "vertical" are both in the eyes of the beholder.

The same calculation as for the tensor product $C \otimes_R D$ which was displayed above shows that $\partial^2 = 0$, so that $\mathrm{Tot}(E)$ is a chain complex. This is no accident: we can make a bicomplex $C\tilde{\otimes}_R D$ with

$$(C\tilde{\otimes}_R D)_{p,q} = C_p \otimes_R D_q$$

in an obvious way, and then

$$C \otimes_R D = \mathrm{Tot}(C\tilde{\otimes}_R D).$$

The notation is awkward: I often write $C \otimes_R D$ for the tensor product of C and D *as well as* for the underlying bicomplex.

Example 10.1. Suppose that C is a chain complex of right R-modules and that N is a left R-module. Then

$$C \otimes_R N = \mathrm{Tot}(C \otimes_R N[0]).$$

Similarly if M is a right R-module and D is a complex of left R-modules, then

$$M \otimes_R D = \mathrm{Tot}(M[0] \otimes_R D).$$

The thing that one usually wants to do with bicomplexes is break them up.

You can already do this with a chain complex C in a trivial way: suppose that F_nC is the subcomplex

$$C_0 \xleftarrow{\partial} \dots \xleftarrow{\partial} C_n \leftarrow 0 \leftarrow 0 \leftarrow \dots$$

For the record, this functor F_nC is sometimes called the *bad truncation* of C, because it doesn't preserve homology groups. Then there is a ascending sequence of inclusion homomorphisms

$$0 = F_{-1}C \to F_0C \to F_1C \to F_2C \to \cdots \to C,$$

and

$$C = \cup_n F_nC = \varinjlim_n F_n(C).$$

This is an elementary example of a *filtration* of the complex C. Observe that

$$H_p(F_nC) = \begin{cases} H_p(C) & \text{if } p < n, \\ Z_n(C) & \text{if } p = n, \text{ and} \\ 0 & \text{if } p > n. \end{cases}$$

There is extra fun that you can have with the short exact sequences

$$0 \to F_{n-1}C \to F_nC \to F_nC/F_{n-1}C \to 0.$$

Specifically (exercise), there is a natural isomorphism

$$F_nC/F_{n-1}C \cong C_n[-n]$$

where $C_n[-n]$ is the chain complex which consists of C_n in degree n and is 0 otherwise.

This last (awkward) notation is meant to be consistent with the *shift operator*. Suppose that D

is a chain complex and $m \in \mathbb{Z}$. Then the *shifted complex* $D[m]$ is the chain complex with

$$D[m]_p = \begin{cases} 0 & \text{if } p+m < 0, \text{ and} \\ D_{p+m} & \text{if } p+m \geq 0. \end{cases}$$

The notation can be confusing: if $m \geq 0$ then $D \mapsto D[-m]$ shifts up ("suspends") m times, while $D \mapsto D[m]$ shifts down (almost "loops") m times.

Observe also that $D[m][n] = D[m+n]$ if m and n have the same parity (ie. $m, n \geq 0$ or $m, n \leq 0$), and that $M[0][-n] = M[-n]$ for a module M.

Exercise 10.2. Show that the formula $D[m][n] = D[m+n]$ can fail if m and n have opposite signs.

Lemma 10.3. *1) Suppose that $m \leq 0$. Show that there are natural isomorphisms*

$$H_p(D[m]) \cong \begin{cases} 0 & \textit{if } p+m < 0, \\ H_{p+m}(D) & \textit{if } p+m \geq 0. \end{cases}$$

2) If $m > 0$, show that

$$H_p(D[m]) \cong \begin{cases} \operatorname{cok}(D_{m+1} \xrightarrow{\partial} D_m) & \textit{if } p = 0, \\ H_{p+m}(D) & \textit{if } p > 0. \end{cases}$$

The proof of this result is an exercise.

The inclusions $F_{n-1}C \to F_nC$ consist of inclusions of split summands (ie. all or nothing) in all degrees, and are therefore preserved by tensor products. Thus if D is a chain complex of left R-modules, then there are short exact sequences

$$0 \to F_{n-1}C \otimes_R D \to F_nC \otimes_R D \to C_n[-n] \otimes_R D \to 0$$

of bicomplexes. The total complex functor Tot is exact (exercise), so there are short exact sequences

$$0 \to F_{n-1}C \otimes_R D \to F_nC \otimes_R D \to \mathrm{Tot}(C_n[-n] \otimes_R D) \to 0$$

of chain complexes. It's easiest just to say what $\mathrm{Tot}(C_n[-n] \otimes_R D)$ is:

$$\mathrm{Tot}(C_n[-n] \otimes_R D)_p = \begin{cases} 0 & \text{if } p < n, \text{ and} \\ C_n \otimes_R D_{p-n} & \text{if } p \geq n, \end{cases}$$

with boundary map $(-1)^n(1 \otimes \partial)$. It follows that there is a canonical isomorphism

$$\mathrm{Tot}(C_n[-n] \otimes_R D) \cong (C_n \otimes_R D)[-n],$$

by an exercise in fiddling with signs. We have proved the following:

Lemma 10.4. *Suppose that C is a complex of right R-modules and that D is a complex of left R-modules, and let $\{F_nC\}$ and $\{F_nD\}$ be the* *natural filtrations of C and D respectively, as defined above. Suppose that $n \geq 0$.*

1) There is a natural isomorphism

$$H_p((F_nC/F_{n-1}C)\otimes_R D) \cong \begin{cases} H_{p-n}(C_n \otimes_R D) & \text{if } p-n \geq 0, \\ 0 & \text{if } p-n < 0. \end{cases}$$

2) There is a natural isomorphism

$$H_p(C\otimes_R(F_nD/F_{n-1}D)) \cong \begin{cases} H_{p-n}(C \otimes_R D_n) & \text{if } p-n \geq 0, \\ 0 & \text{if } p-n < 0. \end{cases}$$

Lemma 10.5. *1) Suppose that P is a cofibrant complex of right R-modules and that the map $g : D \to D'$ is a weak equivalence of chain complexes of left R-modules. Then the induced map of tensor product complexes*

$$1 \otimes g : P \otimes_R D \to P \otimes_R D'$$

is a weak equivalence.

2) Suppose that Q is a cofibrant complex of left R-modules and that $f : C \to C'$ is a weak equivalence of complexes of right R-modules. Then the induced map of tensor products

$$f \otimes 1 : C \otimes_R Q \to C' \otimes_R Q$$

is a weak equivalence.

Proof. We'll prove statement 1). The proof of the second statement is similar.

Let $\{F_nP\}$ be the natural filtration of the complex P which is discussed above. The map $g : D \to D'$ induces a morphism

$$1 \otimes g : F_nP \otimes_R D \to F_nP \otimes_R D'$$

as well as maps

$$1 \otimes g : (F_nP/F_{n-1}P) \otimes_R D \to (F_nP/F_{n-1}P) \otimes_R D'$$

These last maps can be identified up to isomorphism with the morphisms

$$1 \otimes g : (P_n \otimes_R D)[-n] \to (P_n \otimes_R D')[-n],$$

which morphisms are weak equivalences by Lemma 9.14, together with the fact that negative shifts (suspensions) preserve weak equivalences by Lemma 10.3.

An inductive argument (in n) which is based on the comparisons of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F_{n-1}P \otimes_R D & \to & F_nP \otimes_R D & \to & (F_nP/F_{n-1}P) \otimes_R D & \to & 0 \\ & & \downarrow{\scriptstyle 1\otimes g} & & \downarrow{\scriptstyle 1\otimes g} & & \downarrow{\scriptstyle 1\otimes g} & & \\ 0 & \to & F_{n-1}P \otimes_R D' & \to & F_nP \otimes_R D' & \to & (F_nP/F_{n-1}P) \otimes_R D' & \to & 0 \end{array}$$

finishes the proof. □

Suppose that C is a complex of right R-modules and that D is a complex of left R-modules. Choose natural cofibrant models $\pi_C : P_C \to C$ and $\pi_D : Q_D \to D$. The *derived tensor product* $C\hat{\otimes}_R D$ is defined by

$$C\hat{\otimes}_R D = P_C \otimes_R Q_D.$$

This construction is functorial in C and D. Here are some salient features:

Corollary 10.6. *1) Any weak equivalence $f : C \to C'$ induces a weak equivalence*

$$f\hat{\otimes}1 : C\hat{\otimes}_R D \to C'\hat{\otimes}_R D.$$

2) Any weak equivalence $g : D \to D'$ induces a weak equivalence

$$1\hat{\otimes}g : C\hat{\otimes}D \to C\hat{\otimes}D'$$

3) Suppose that $p : P \to C$ is a cofibrant model of C (ie. weak equivalence with P cofibrant) and that $q : Q \to D$ is a cofibrant model of D. Then the complexes $P \otimes_R D$ and $C \otimes_R Q$ are weakly equivalent to $C\hat{\otimes}_R D$.

Proof. For statement 1), if f is a weak equivalence then the induced map $f_* : P_C \to P_{C'}$ is a weak equivalence, so the map

$$f\hat{\otimes}1 := f_* \otimes 1 : P_C \otimes_R Q_D \to P_{C'} \otimes_R Q_D$$

is a weak equivalence by Lemma 10.5. Statement 2) has a similar proof.

For statement 3), find liftings

$$\begin{array}{ccc} & & P_C \\ & \overset{\theta}{\nearrow} & \downarrow \pi_C \\ P & \xrightarrow[p]{} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} & & Q_D \\ & \overset{\gamma}{\nearrow} & \downarrow \pi_D \\ Q & \xrightarrow[q]{} & D \end{array}$$

Then the maps θ and γ are weak equivalences, and there are induced weak equivalences

$$P \otimes_R D \xleftarrow{1 \otimes \pi_D} P \otimes_R Q_D \xrightarrow{\theta \otimes 1} P_C \otimes_R Q_D,$$

and

$$C \otimes_R Q \xleftarrow{\pi_C \otimes 1} P_C \otimes_R Q \xrightarrow{1 \otimes \gamma} P_C \otimes_R Q_D$$

by Lemma 10.5. □

The *higher torsion products* $\mathrm{Tor}_n(C, D)$ are defined to be the homology groups

$$\mathrm{Tor}_n(C, D) = H_n(C \hat{\otimes} D)$$

of the derived tensor product. These groups are functorial in both C and D.

Remark 10.7. 1) In view of Corollary 10.6, there is quite a bit of flexibility in computing these higher torsion product groups up to isomorphism, since there are induced isomorphisms

$$\mathrm{Tor}_n(C, D) \cong H_n(P \otimes_R D) \cong H_n(C \otimes_R Q),$$

where $p : P \to C$ and $q : Q \to D$ are cofibrant replacement of C and D.

2) Suppose that M is a right R-module and that N is a left R-module. Then there is a natural isomorphism

$$\mathrm{Tor}_n(M,N) \cong H_n(M[0]\hat{\otimes}_R N[0]) = \mathrm{Tor}_n(M[0],N[0]).$$

In effect, if $p : P \to M[0]$ is a cofibrant model (projective resolution) then there is an isomorphism

$$\mathrm{Tor}_n(M[0],N[0]) \cong H_n(P \otimes_R N[0]),$$

and there is an isomorphism of complexes

$$P \otimes_R N[0] \cong P \otimes_R N.$$

Similarly, if $q : Q \to N[0]$ is a cofibrant model then there is an isomorphism

$$\mathrm{Tor}_n(M[0],N[0]) \cong H_n(M \otimes_R Q).$$

Compare with Lemma 9.15.

The higher torsion products $\mathrm{Tor}_n(C,D)$ can be a bit difficult to compute. Generally, they sit in a spectral sequence, which is a computational gadget that will be discussed later. In the interim, here's something that's nice to know:

Lemma 10.8. *Suppose that C is a complex of right R-modules and that D is a complex of left*

R-modules and that $m, n \geq 0$. Suppose that $H_i(C) = 0$ for $i \leq m$ and $H_j(D) = 0$ for $j \leq n$ then $\operatorname{Tor}_k(C, D) = 0$ *for $k \leq m + n + 1$.*

Proof. There is a exact sequence of chain complexes

$$0 \to f_n C \xrightarrow{i} C \xrightarrow{p} P_n C \to 0$$

where $P_n C_k = 0$ for $k \geq n + 2$, $P_n C_k = C_k$ for $k \leq n$, and $P_n C_{n+1} = B_n(C)$ with boundary $\partial : P_n C_{n+1} \to P_n C_n$ defined by the inclusion $B_n(C) \to C_n$. The map p induces homology isomorphisms

$$p_* : H_k(C) \xrightarrow{\cong} H_k(P_n C)$$

for $k \leq n$ while the inclusion i of the kernel of p induces isomorphisms

$$i_* : H_k(f_n C) \to H_k(C)$$

for $k \geq n + 1$. Observe as well that $f_n C_k = 0$ for $k \leq n$.

Under the assumptions of the Lemma, the maps $i : f_m C \to C$ and $i : f_n D \to D$ are weak equivalences. The higher torsion products are invariants of weak equivalences in C and D, so we can assume that $C_i = 0$ for $i \leq m$ and that $D_j = 0$ for $j \leq n$.

The complex $C[m+1]$ (with C_{m+1} in degree 0) has a cofibrant resolution $\pi : P \to C[m+1]$, and suspending the weak equivalence π gives a weak equivalence $\pi_* : P[-m-1] \to C$. All of the modules making up the complex $P[-m-1]$ are projective, so that $P[-m-1]$ is cofibrant. We can therefore assume that C has a cofibrant model $p : P' \to C$ such that $P_i = 0$ for $i \leq m$.

The chain complex $P \otimes_R D$ satisfies

$$(P \otimes_R D)_k = 0$$

for $k \leq m+n+1$. In effect if $i+j < m+n+2$ then $i < m+1$ or $j < n+1$. Thus,

$$\mathrm{Tor}_k(C, D) \cong H_k(P \otimes_R D) = 0$$

if $k \leq m+n+1$. □

Remark 10.9. The object P_nC is called, variously, the n^{th} *Postnikov section* of the complex C, or the *good truncation* of C at level n. The functor $C \mapsto P_nC$ is a good truncation because it preserves weak equivalences. The "Postnikov section" term is a homotopy theory thing, and is consistent with corresponding constructions for spaces and spectra.

Corollary 10.10. *Suppose that C is a complex of right R-modules and that D is a complex of left R-modules. Then the natural maps $C \to H_0(C)[0]$ and $D \to H_0(D)[0]$ induce an isomorphism*

$$\begin{aligned}\mathrm{Tor}_0(C, D) &\cong \mathrm{Tor}_0(H_0(C)[0], H_0(D)[0]) \\ &\cong H_0C \otimes_R H_0D.\end{aligned}$$

Proof. The chain complex morphism $C \to H_0(C)[0]$ is surjective in all degrees and has a kernel K such that $H_0(K) = 0$.

Suppose that $q : Q \to D$ is a cofibrant model of D. Then the short exact sequence

$$0 \to K \otimes_R Q \to C \otimes_R Q \to H_0(C)[0] \otimes_R Q \to 0$$

induces an exact sequence

$$\dots \xrightarrow{\partial} \mathrm{Tor}_0(K, D) \to \mathrm{Tor}_0(C, D) \to \mathrm{Tor}_0(H_0(C)[0], D) \to 0$$

and $\mathrm{Tor}_0(K, D) = 0$ by Lemma 10.8. It follows that the canonical map $C \to H_0(C)[0]$ induces an isomorphism

$$\mathrm{Tor}_0(C, D) \xrightarrow{\cong} \mathrm{Tor}_0(H_0(C)[0], D)$$

for all complexes D. By a similar argument, the map $D \to H_0(D)[0]$ induces an isomorphism

$$\mathrm{Tor}_0(H_0(C)[0], D) \xrightarrow{\cong} \mathrm{Tor}_0(H_0(C)[0], H_0(D)[0])$$

We have already seen in Remark 10.7 and Lemma 9.9 that are isomorphisms

$$\begin{aligned}\mathrm{Tor}_0(H_0(C)[0], H_0(D)[0]) &\cong \mathrm{Tor}_0(H_0(C), H_0(D)) \\ &\cong H_0(C) \otimes_R H_0(D).\end{aligned}$$

□

Sometimes, you just get lucky:

Theorem 10.11 (Künneth)**.** *Suppose that C and D are chain complexes of abelian groups. Then there are short exact sequences*

$$0 \longrightarrow (H_*(C) \otimes H_*(D))_n \longrightarrow H_n(C \hat{\otimes} D) \longrightarrow \mathrm{Tor}(H_*(C), H_*(D))_{n-1} \longrightarrow 0$$

Here, for the sake of notational convenience, we set

$$(H_*(C) \otimes H_*(D))_n := \bigoplus_{p+q=n} H_p(C) \otimes H_q(D)$$

and

$$\mathrm{Tor}(H_*(C), H_*(D))_{n-1} := \bigoplus_{r+s=n-1} \mathrm{Tor}(H_r(C), H_s(D)).$$

Proof. By the usual small object argument, there is a natural cofibrant model $F \to C$ such that F is a complex which is free abelian in all degrees. We can therefore assume that C is a complex of free abelian groups.

The submodules $Z_n(C)$ and $B_n(C)$ of C_n are free abelian (since subgroups of free abelian groups are free abelian), and the short exact sequence

$$0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$$

gives a projective resolution $P^n \to H_0(C)[0]$ of the homology group $H_n(C)$. The lift exists in the diagram

$$\begin{array}{ccc} B_n(C) & \overset{\theta^n_{n+1}}{\dashrightarrow} & C_{n+1} \\ \downarrow & & \downarrow \partial \\ Z_n(C) & \underset{\theta^n_n}{\longrightarrow} & C_n \end{array}$$

where θ^n_n is the usual inclusion, and there is a chain map $\theta^n : P^n[-n] \to C$. The map θ^n induces an isomorphism

$$H_n(P^n[-n]) \cong H_n(C)$$

in degree n. All maps θ^n together induce a weak equivalence

$$\sum \theta_n : \bigoplus_{n \geq 0} P_n \to C.$$

Suppose that $Q \to D$ is a cofibrant model for D. Suppose that $r \leq n$. Then there are weak equivalences

$$\begin{array}{ccc}
P^r[-r] \otimes D & \xleftarrow{\simeq} & P^r[-r] \otimes Q \\
& & \downarrow \simeq \\
& & H_r(C)[-r] \otimes Q \xrightarrow[\cong]{} (H_r(C) \otimes Q)[-r].
\end{array}$$

It therefore follows from the universal coefficients theorem that there is a short exact sequence

$$\begin{array}{c}
0 \longrightarrow H_r(C) \otimes H_{n-r}(D) \longrightarrow H_n(P^r[-r] \otimes D) \\
\downarrow \\
\mathrm{Tor}(H_r(C), H_{n-r-1}(D)) \longrightarrow 0
\end{array}$$

form each $r \leq n$. The direct sum of these exact sequences, indexed over $0 \leq r \leq n$ is the exact sequence in the statement of the Theorem. $\square$