Contents

17 Proper model structures

$\mathbf{M}$ is a fixed closed model category, for a while.

Here's a basic principle:

Lemma 17.1. *Suppose $f : X \to Y$ is a morphism of $\mathbf{M}$, with both X and Y cofibrant.*

Then f has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, "u"] \\ & Y \end{tikzcd}$$

such that i is a cofibration, and u is a weak equivalence which is left inverse to a trivial cofibration $j : Y \to Z$.

Proof. The construction is an abstraction of the classical mapping cylinder. It is dual to the replacement of a map between fibrant objects by a fibration (Section 13). $\square$

Lemma 17.2. *Suppose given a pushout diagram*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] & B \arrow[d] \\
C \arrow[r, "u_*"'] & D
\end{tikzcd}$$

in $\mathbf{M}$ *with all objects cofibrant, i a cofibration and u a weak equivalence.*

Then u_* is a weak equivalence.

To put it a different way, in the category of cofibrant objects in a model category $\mathbf{M}$, the class of weak equivalences is closed under pushout along cofibrations.

Proof. By Lemma 17.1, and since trivial cofibrations are closed under pushout, it suffices to assume that there is a trivial cofibration $j : B \to A$ with $uj = 1_B$.

Form the diagram

$$\begin{tikzcd}
B \arrow[r, "j"] \arrow[d, "j"'] & A \arrow[r, "u"] \arrow[dd, "j_*"'] \arrow[dddr, "i"] & B \arrow[dd] \arrow[dddr] & \\
A \arrow[d, "i"'] & & & \\
C \arrow[r, "\tilde{j}"] \arrow[drr, "1_C"'] & \tilde{D} \arrow[r, "\tilde{u}"] \arrow[dr, "f"] & B_* \arrow[dr, "f_*"] & \\
& & C \arrow[r, "u_*"'] & D
\end{tikzcd}$$

in which the two back squares are pushouts.

j is a trivial cofibration so $\tilde{j}$ is a trivial cofibration, and so $\tilde{u}$ is a weak equivalence (since $\tilde{u}\tilde{j}$ is an isomorphism). f is a weak equivalence, so it suffices to show that the map f_* is a weak equivalence.

f_* is a map between cofibrant objects of the model category $B/\mathbf{M}$ which is obtained by pushing out the map $j_* \xrightarrow{f} i$ of $A/\mathbf{M}$ along u.

The pushout functor takes trivial cofibrations of slice categories to trivial cofibrations, thus preserves weak equivalences between cofibrant objects. $\square$

Remark: For the last proof, you need to know (exercise) that if $\mathbf{M}$ is a model category and A is an object of $\mathbf{M}$, then the slice category $A/\mathbf{M}$ has a model structure for which a morphism

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[rr, "f"'] & & C \end{tikzcd}$$

is a weak equivalence (respectively cofibration, fibration) if and only if the map $f : B \to C$ is a weak equivalence (respectively cofibration, fibration) of $\mathbf{M}$.

The dual structure on the slice category $\mathbf{M}/A$ has a similar description.

Here is the dual of Lemma 17.2:

Lemma 17.3. *Suppose given a pullback diagram*

$$\begin{tikzcd} W \arrow[r, "u_*"] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r, "u"'] & Y \end{tikzcd}$$

in $\mathbf{M}$*, with all objects fibrant, p a fibration and u a weak equivalence.*

Then u_* is a weak equivalence.

Thus, in the category of fibrant objects in $\mathbf{M}$ the class of weak equivalences is closed under pullback along fibrations.

Definition 17.4. A model category $\mathbf{M}$ is

1) *right proper* if the class of weak equivalences is closed under pullback along fibrations,
2) *left proper* if the class of weak equivalences is closed under pushout along cofibrations,
3) *proper* if it is both right and left proper.

Examples: 1) The category $s\mathbf{Set}$ is proper.

All simplicial sets are cofibrant, so $s\mathbf{Set}$ is left proper. Given a pullback

$$\begin{tikzcd} W \arrow[r, "u_*"] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r, "u"'] & Y \end{tikzcd}$$

in $s\mathbf{Set}$ with p a fibration and u a weak equivalence, the induced diagram

$$\begin{tikzcd} |W| \arrow[r, "|u_*|"] \arrow[d] & |X| \arrow[d, "|p|"] \\ |Z| \arrow[r, "|u|"'] & |Y| \end{tikzcd}$$

of spaces is a pullback (realization is exact) in which $|p|$ is a Serre fibration (Quillen's theorem: Theorem 13.1) and $|u|$ is a weak equivalence. All spaces are fibrant, so $|u_*|$ is a weak equivalence by Lemma 17.3, and so u_* is a weak equivalence of $s\mathbf{Set}$.

2) All spaces are fibrant, so **CGWH** is right proper by Lemma 17.3. This category is also left proper by (non-abelian) *excision*, and the fact that $s\mathbf{Set}$ is left proper.

The excision statement is the following:

Lemma 17.5. *Suppose the open subsets U_1, U_2 cover a space Y.*

Then the induced map

$$S(U_1) \cup_{S(U_1 \cap U_2)} S(U_2) \to S(Y)$$

is a weak equivalence of simplicial sets.

Lemma 17.5 can be proved with simplicial approximation techniques [3].

3) The categories of simplicial groups and simplicial modules are right proper. The category of simplicial modules is also left proper (exercise).

4) There is a model structure on $s\mathbf{Set}$ for which the cofibrations are the monomorphisms, and the weak equivalences are those maps $X \to Y$ which induce rational homology isomorphisms

$$H_*(X, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$$

(this is the rational homology local model structure — it is one of the objects of study of rational homotopy theory).

There is a pullback square

$$\begin{tikzcd}
K(\mathbb{Q}/\mathbb{Z}, 0) \arrow[r, "u_*"] \arrow[d] & P \arrow[d, "p"] \\
K(\mathbb{Z}, 1) \arrow[r, "u"'] & K(\mathbb{Q}, 1)
\end{tikzcd}$$

where p is a fibration, P is contractible, and u is induced by the inclusion $\mathbb{Z} \subset \mathbb{Q}$. The map u is a rational homology isomorphism since $\mathbb{Q}/\mathbb{Z}$ consists of torsion groups, while u_* is not.

Here's the **glueing lemma**:

Lemma 17.6. *Suppose given a commutative cube*

$$
\begin{tikzcd}
A_1 \arrow[rr, "j_1"] \arrow[dr, "i_1"] \arrow[dd, "f_A"'] & & B_1 \arrow[dr] \arrow[dd, "f_B" near start] & \\
& C_1 \arrow[rr, crossing over] & & D_1 \arrow[dd, "f_D"] \\
A_2 \arrow[rr, "j_2" near end] \arrow[dr, "i_2"'] & & B_2 \arrow[dr] & \\
& C_2 \arrow[rr] \arrow[from=uu, crossing over, "f_C" near start] & & D_2
\end{tikzcd}
$$

in which all objects are cofibrant, i_1 and i_2 are cofibrations, the top and bottom faces are pushouts, and the maps f_A, f_B and f_C are weak equivalences.

Then f_D is a weak equivalence.

Proof. By Lemma 17.2, it suffices to assume that the maps j_1 and j_2 are cofibrations.

Form the diagram

$$\begin{tikzcd}
A_1 \arrow[rr, "j_1"] \arrow[dr, "i_1"] \arrow[dddd, "f_A"'] & & B_1 \arrow[dr] \arrow[dd, "f_{A*}"] & \\
& C_1 \arrow[rr, "j_{1*}"] \arrow[dddd, "f_C"'] & & D_1 \arrow[dd, "f_{C*}"] \\
& & B' \arrow[dr, "i_{2*}"] \arrow[dd, "\eta_B"'] & \\
& & & D' \arrow[dd, "\eta_D"] \\
A_2 \arrow[uurr] \arrow[rr, "j_2"] \arrow[dr, "i_2"'] & & B_2 \arrow[dr] & \\
& C_2 \arrow[uurr] \arrow[rr] & & D_2
\end{tikzcd}$$

in which f_{A*} is the pushout of f_A along j_1 and f_{C*} is the pushout of f_C along j_{1*}.

All squares in the prism are pushouts, i_{2*} is a cofibration, and η_B is a weak equivalence. It follows from Lemma 17.2 that η_D is a weak equivalence.

f_{C*} is also a weak equivalence, so f_D is a weak equivalence. $\square$

Remarks:

1) Lemma 17.6 has a dual, which is usually called the **coglueing lemma**.

2) The statement of Lemma 17.6 holds in any left proper model category, by the same argument, while its dual holds in any right proper model category.

Let's be explicit. Here's the cogluing lemma for right proper model categories:

Lemma 18.1. *Suppose* **M** *is right proper model. Suppose given a diagram*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "\simeq"'] & Y_1 \arrow[d, "\simeq"] & Z_1 \arrow[l, "p_1"'] \arrow[d, "\simeq"] \\ X_2 \arrow[r] & Y_2 & Z_2 \arrow[l, "p_2"] \end{tikzcd}$$

for which the vertical maps are weak equivalences and the maps p_1, p_2 are fibrations.

Then the map

$$X_1 \times_{Y_1} Z_1 \to X_2 \times_{Y_2} Z_2$$

is a weak equivalence.

The model category **M** will be right proper throughout this section.

A commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd} \tag{1}$$

in **M** is **homotopy cartesian** if f has a factorization

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "f"'] & U \arrow[d, "p"] \\ & Y \end{tikzcd} \tag{2}$$

such that p is a fibration and θ is a weak equivalence, and such that the induced map

$$W \xrightarrow{\theta_*} Z \times_Y U$$

is a weak equivalence.

Slogan 1: The choice of factorization of f doesn't matter.

Lemma 18.2. *Suppose given a second factorization*

$$\begin{tikzcd} X \arrow[r, "\theta'"] \arrow[dr, "f"'] & U' \arrow[d, "p'"] \\ & Y \end{tikzcd}$$

of the map f in the commutative square (1). with θ' a weak equivalence and p' a fibration. Then the map

$$W \xrightarrow{\theta_*} Z \times_Y U$$

is a weak equivalence if and only if the map

$$W \xrightarrow{\theta'_*} Z \times_Y U'$$

is a weak equivalence.

Proof. It suffices to assume that the maps θ and θ' are trivial cofibrations. To see this, factorize θ as

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\theta"'] & V \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

where π is a trivial fibration and i is a trivial cofibration. Then in the diagram

$$\begin{tikzcd} W \arrow[r, "i_*"] \arrow[dr, "\theta_*"'] & Z \times_Y V \arrow[d, "\pi_*"] \\ & Z \times_Y U \end{tikzcd}$$

the map π_* is a trivial fibration, so θ_* is a weak equivalence if and only if i_* is a weak equivalence.

Now suppose θ and θ' are trivial cofibrations. Then the lifting s exists in the diagram

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "\theta'"'] & U \arrow[d, "p"] \\ U' \arrow[r, "p'"'] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

and the induced map s_* in the diagram

$$\begin{tikzcd} W \arrow[r, "\theta'_*"] \arrow[dr, "\theta_*"'] & Z \times_Y U' \arrow[d, "s_*"] \\ & Z \times_Y U \end{tikzcd}$$

is a weak equivalence by Lemma 18.1. Thus, θ_* is a weak equivalence if and only if θ'_* is a weak equivalence. $\square$

Slogan 2: It doesn't matter whether you factorize f or g.

Lemma 18.3. *Suppose*

$$\begin{tikzcd} Z \arrow[r, "\gamma"] \arrow[dr, "g"'] & V \arrow[d, "q"] \\ & Y \end{tikzcd}$$

is a factorization of the map g in the diagram (1) with q a fibration and γ a weak equivalence, and $f = p \cdot \theta$ with p a fibration and θ a weak equivalence as in (2). Then the map $\theta_* : W \to Z \times_Y U$ is a weak equivalence if and only if the map $\gamma_* : W \to V \times_Y X$ is a weak equivalence.

Proof. There is a commutative square

$$\begin{tikzcd} W \arrow[r, "\theta_*"] \arrow[d, "\gamma_*"'] & Z \times_Y U \arrow[d, "\gamma_*", "\simeq"'] \\ V \times_Y X \arrow[r, "\simeq", "\theta_*"'] & V \times_Y U \end{tikzcd}$$

The indicated maps are weak equivalences since they are pull backs of weak equivalences along fibrations. $\square$

The following is a rephrasing of the homotopy cogluing lemma for a right proper model category $\mathbf{M}$:

Lemma 18.4. *Suppose given a commutative cube*

$$\begin{tikzcd}[row sep=small, column sep=small]
W_1 \arrow[rr] \arrow[dr] \arrow[dd, "f_W"'] & & X_1 \arrow[dr] \arrow[dd, "f_X" near start] & \\
& Z_1 \arrow[rr, crossing over] & & Y_1 \arrow[dd, "f_Y"] \\
W_2 \arrow[rr] \arrow[dr] & & X_2 \arrow[dr] & \\
& Z_2 \arrow[rr] \arrow[from=uu, crossing over, "f_Z" near start] & & Y_2
\end{tikzcd}$$

in a right proper model category $\mathbf{M}$ *such that the top and bottom faces are homotopy cartesian, and the vertical maps f_Z, f_X and f_Y are weak equivalences.*

Then f_W is a weak equivalence.

This result follows from the dual of Lemma 17.6.

Homotopy cartesian diagrams behave much like pullback diagrams:

Lemma 18.5. *Suppose* $\mathbf{M}$ *is right proper.*

1) Suppose given a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha"] \arrow[d] & X_2 \arrow[d] \\
Y_1 \arrow[r, "\beta"'] & Y_2
\end{tikzcd}$$

in $\mathbf{M}$ *such that the maps* α *and* β *are weak equivalences. Then this diagram is homotopy cartesian.*

2) *Suppose given a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{II}"] & X_3 \arrow[d] \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \end{tikzcd}$$

Then

a) *if the squares* $\mathbf{I}$ *and* $\mathbf{II}$ *are homotopy cartesian, then the composite square* $\mathbf{I}+\mathbf{II}$ *is homotopy cartesian,*

b) *if* $\mathbf{I}+\mathbf{II}$ *and* $\mathbf{II}$ *is homotopy cartesian then* $\mathbf{I}$ *is homotopy cartesian.*

Proof. The proof is an (important) exercise. □

A **homotopy fibre sequence** (or just **fibre sequence**) is a homotopy cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ P \arrow[r] & Y \end{tikzcd}$$

in which P is contractible (ie. weakly equivalent to the terminal object). F is a **homotopy fibre** of the map f.

Remark: All concepts and results of this section have duals in left proper model categories, where one has homotopy cocartesian diagrams, homotopy cofibre sequences, and homotopy cofibres.

19 Diagrams of spaces

Suppose I is a small category. $s\mathbf{Set}^I$ denotes the category of functors $I \to s\mathbf{Set}$ and their natural transformations. $s\mathbf{Set}^I$ is a **diagram category**.

Some people say that it is the category of *simplicial presheaves* on the category I.

$s\mathbf{Set}^I$ is the category of simplicial sheaves for the chaotic topology on I (which means no topology at all).

A map (natural transformation) $f : X \to Y$ of I-diagrams is a **weak equivalence** (sometimes called a *sectionwise weak equivalence* or *pointwise weak equivalence*) if all maps $f : X(i) \to Y(i)$, $i \in I$, are weak equivalences of simplicial sets.

There are many model structures on the diagram category $s\mathbf{Set}^I$ for which the weak equivalences are as described, but I will single out two of them:

- The **projective structure**: The fibrations are defined sectionwise: a *projective fibration* is a map $p : X \to Y$ for which consists of Kan fibrations $f : X(i) \to Y(i)$, $i \in I$, in sections. A *projective cofibration* is a map which has the left lifting property with respect to all trivial projective fibrations.

- The **injective structure**: The cofibrations are defined sectionwise. A *cofibration* of I-diagrams is a monomorphism of $s\mathbf{Set}^I$, and an *injective fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

The projective structure was introduced by Bousfield and Kan [1], and is easy to construct.

The ***i*-sections functor** $X \mapsto X(i)$ has a left adjoint L_i with

$$L_i(K) = \hom(i,\ ) \times K$$

for simplicial sets K.

A map $p : X \to Y$ of $s\mathbf{Set}^I$ is a projective fibration (respectively projective trivial fibration) if and only if it has the right lifting property with respect to the set of all maps $L_i(\Lambda^n_k) \to L_i(\Delta^n)$ (repectively with respect to the set of maps $L_i(\partial\Delta^m) \to L_i(\Delta^m)$.

The factorization axiom is proved by standard small object arguments, **CM4** is proved by the usual tricks, and the rest of the axioms are trivial.

Note that we have specified generating sets for the trivial projective cofibrations and the projective cofibrations.

To summarize:

Lemma 19.1. *The sectionwise weak equivalences, projective fibrations and projective cofibrations give the diagram category $s\mathbf{Set}^I$ the structure of a proper closed simplicial model category. This model structure is cofibrantly generated.*

Heller [2] is credited with the introduction of the injective structure on $s\mathbf{Set}^I$. It is also a special case of the model structure for simplicial sheaves which first appeared in Joyal's seminal letter to Grothendieck [4].

The injective structure is a little trickier to derive. Pick an infinite cardinal $\alpha > |\mathrm{Mor}(I)|$. Then one must prove a bounded cofibration condition:

Lemma 19.2. *Given a trivial cofibration $X \to Y$ and an α-bounded subobject $A \subset Y$ there is an α-bounded B with $A \subset B \subset Y$ such that $B \cap X \to B$ is a trivial cofibration.*

An I-diagram A is **α-bounded** if $|A(i)| < \alpha$ for all $i \in I$, and a cofibration $A \to B$ is α-bounded if B is α-bounded.

It follows (see the proof of Lemma 11.5 (Lecture 04)) that a map $p : X \to Y$ of $s\mathbf{Set}$ is an injective fibration (respectively trivial injective fibration) if and only if it has the right lifting property with respect to all α-bounded trivial cofibrations (respectively with respect to all α-bounded cofibrations).

The factorization axiom **CM5** for the injective structure follows from a transfinite small object argument — see the proof of Lemma 11.4. The lifting axiom **CM4** also follows, while the remaining axioms **CM1** — **CM3** are trivial.

We have "proved":

Theorem 19.3. *The sectionwise weak equivalences, cofibrations and injective fibrations give the category $s\mathbf{Set}^I$ the structure of a proper closed simplicial model category. This model structure is cofibrantly generated.*

For I-diagrams X and Y, write $\mathbf{hom}(X,Y)$ for the simplicial set whose set of n-simplices is the collection of maps $X \times \Delta^n \to Y$ (here Δ^n is identified with a constant I-diagram). For a simplicial set K and I-diagram X, the I-diagram X^K is specified at objects $i \in I$ by

$$X^K(i) = \mathbf{hom}(K, X(i)).$$

There is also an I-diagram $X \otimes K := X \times K$ given by

$$(X \times K)(i) = X(i) \times K.$$

If $i : A \to B$ is a cofibration (respectively projective cofibration) and $j : K \to L$ is a cofibration of simplicial sets, then the map

$$(i, j) : (B \times K) \cup (A \times L) \subset B \times L$$

is a cofibration (respectively projective cofibration) which is trivial if either i or j is trivial. The only issue with this is in showing that (i, j) is projective if i is projective, but it's true for generators $L_k(A') \to L_k(B')$, so it's true.

Finally, every projective cofibration is a cofibration, and every injective fibration is a projective fibration.

It follows that weak equivalences are stable under pullback along injective fibrations, and weak equivalences are stable under pushout along projective cofibrations, by properness for simplicial sets.

A model category **M** is **cofibrantly generated** if there is a set I of trivial cofibrations and a set J of cofibrations such that a map p is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all members of I (respectively J).

Exercise: Fill in the blanks in the proofs of Lemma 19.1 and Theorem 19.3.

20 Homotopy limits and colimits

The constant functor $\Gamma : s\mathbf{Set} \to s\mathbf{Set}^I$ has both a right and left adjoint, given by limit and colimit, respectively.

Specifically,

$$\Gamma(X)(i) = X.$$

and all maps $i \to j$ of I are sent to 1_X.

Γ preserves weak equivalences and cofibrations, and takes fibrations to projective fibrations.

The colimit functor

$$\varinjlim : s\mathbf{Set}^I \to s\mathbf{Set}$$

therefore takes projective cofibrations to cofibrations and takes trivial projective cofibrations to trivial cofibrations.

Homotopy colimits

The adjunction

$$\varinjlim : s\mathbf{Set}^{I} \leftrightarrows s\mathbf{Set} : \Gamma$$

forms a Quillen adjunction for the projective structure on $s\mathbf{Set}^{I}$.

The **homotopy left derived functor** $L\varinjlim$ is defined by

$$L\varinjlim(X) = \varinjlim\, Y,$$

where $Y \to X$ is a weak equivalence with Y projective cofibrant.

Y is a *projective cofibrant replacement* (or *projective cofibrant resolution*, or *projective cofibrant model*) of X.

The functor $X \mapsto \varinjlim X$ takes trivial projective cofibrations to trivial cofibrations, hence takes weak equivalences between projective cofibrant objects to weak equivalences.

The homotopy type of $L\varinjlim(X)$ is independent of the choice of projective cofibrant resolution for X.

The object $L\varinjlim(X)$ has another name: it's called the **homotopy colimit** for the diagram X, and one writes

$$\underrightarrow{\text{holim}}\, X = L\varinjlim(X) = \varinjlim Y,$$

where $Y \to X$ is a projective cofibrant model.

Examples:

1) Consider all diagrams

$$B \leftarrow A \to C$$

of simplicial sets. This diagram is projective cofibrant if and only if all displayed morphisms are cofibrations (exercise). Every diagram

$$Z \xleftarrow{f} X \xrightarrow{g} Y$$

has a resolution by a diagram of cofibrations.

Thus, to form the homotopy pushout of f and g, replace f and g by cofibrations i and j, as in

$$\begin{tikzcd} B \arrow[d, "\simeq"'] & & C \arrow[d, "\simeq"] \\ Z & X \arrow[ul, "j"'] \arrow[ur, "i"] \arrow[l, "f"] \arrow[r, "g"'] & Y \end{tikzcd}$$

and then the homotopy pushout is $B \cup_X C$.

By (left) properness, you only need to replace one of f or g: there are weak equivalences

$$B \cup_X Y \xleftarrow{\simeq} B \cup_X C \xrightarrow{\simeq} Z \cup_X C.$$

Thus, any homotopy co-cartesian diagram constructs the homotopy pushout.

2) All discrete diagrams are projective cofibrant, so homotopy coproducts and coproducts coincide.

3) Consider all countable diagrams

$$X :\; X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \dots$$

Such a diagram is projective cofibrant if and only if all α_i are cofibrations.

If the comparison

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \dots \\ \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq & & \\ X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \dots \end{array}$$

is a projective cofibrant resolution of X, then the induced map

$$\varinjlim_n A_n \to \varinjlim_n X_n$$

is a weak equivalence by comparing homotopy groups. It follows that the canonical map

$$\underrightarrow{\mathrm{holim}}\; X = \varinjlim A \to \varinjlim X$$

is a weak equivalence.

Homotopy limits

The inverse limit functor

$$\varprojlim : s\mathbf{Set}^I \to s\mathbf{Set}$$

takes injective fibrations to fibrations and takes trivial injective fibrations to trivial fibrations.

The adjunction

$$\Gamma : s\mathbf{Set} \leftrightarrows s\mathbf{Set}^I : \varprojlim$$

forms a Quillen adjunction for the injective structure on $s\mathbf{Set}^I$.

The **homotopy right derived functor** $R\varprojlim$ is defined by

$$R\varprojlim(X) = \varprojlim Z$$

where $\alpha : X \to Z$ is an injective fibrant model for X (ie. α is a sectionwise weak equivalence with Z injective fibrant).

The functor $Z \mapsto \varprojlim Z$ takes trivial injective fibrations to weak equivalences, and therefore takes weak equivalences between injective fibrant objects Z to weak equivalences.

The homotopy type of $R\varprojlim(X)$ is independent the choice of injective fibrant model for X.

The object $R\varprojlim(X)$ is the **homotopy inverse limit** of the diagram X, and one writes

$$\underleftarrow{\mathrm{holim}}\, X = R\varprojlim(X) = \varprojlim\, Z$$

where $X \to Z$ is an injective fibrant model for X.

Examples:

1) A diagram

$$X \xrightarrow{p} Y \xleftarrow{q} Z$$

of simplicial sets is injective fibrant if and only if Y is fibrant and p and q are fibrations.

Suppose given a diagram

$$X_1 \xrightarrow{f} X_2 \xleftarrow{g} X_3$$

and form an injective fibrant model

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "j_1"'] & X_2 \arrow[d, "j_2"] & X_3 \arrow[l, "g"'] \arrow[d, "j_3"] \\ Z_1 \arrow[r, "p"'] & Z_2 & Z_3 \arrow[l, "q"] \end{tikzcd}$$

by choosing a fibrant model j_2 and then factorizing both $j_2 f$ and $j_2 g$ as a trivial cofibration followed by a fibration.

Factorize g as $g = \pi \cdot j$ where j is a trivial cofibration and π is a fibration. There is a lifting

$$\begin{tikzcd} X_3 \arrow[r, "j_3"] \arrow[d, "j"'] & Z_3 \arrow[d, "q"] \\ X_3' \arrow[r, "j_2 \pi"'] \arrow[ur, dotted] & Z_2 \end{tikzcd}$$

There is a comparison diagram

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "j_1"'] & X_2 \arrow[d, "j_2"] & X_3' \arrow[l, "\pi"'] \arrow[d] \\ Z_1 \arrow[r, "p"'] & Z_2 & Z_3 \arrow[l, "q"] \end{tikzcd}$$

in which the vertical maps are weak equivalence and π and q are fibrations. The induced map

$$X_1 \times_{X_2} X_3' \to Z_1 \times_{Z_2} Z_3$$

is a weak equivalence by coglueing (Lemma 18.4).

Every homotopy cartesian diagram of simplicial sets computes the homotopy pullback.

2) A discrete diagram $\{X_i\}$ in $s\mathbf{Set}$ is injective fibrant if and only if all objects X_i are fibrant. The homotopy product of a diagram $\{Y_i\}$ is constructed by taking fibrant replacements $Y_i \to X_i$ for all i, and then forming the product $\prod_i X_i$.

This construction is serious: consider the simplicial sets A_n, $n \geq 1$, where A_n is the string of n copies of Δ^1

$$0 \to 1 \to 2 \to \cdots \to n$$

glued end to end.

Each A_n is weakly equivalent to a point so their homotopy product is contractible, but $\prod_{n \geq 1} A_n$ is not path connected.

3) A countable diagram (aka. a "tower")

$$X : X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$$

is injective fibrant if and only if X_1 is fibrant and all morphisms in the tower are fibrations.

The long exact sequences associated to the fibrations in the tower entangle to define a spectral sequence (the Bousfield-Kan spectral sequence [1]) which computes the homotopy groups of $\varprojlim X_n$, at least in good cases.

References

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