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24 Bisimplicial abelian groups, derived functors

Homology

Suppose $A : I \rightarrow \mathbf{Ab}$ is a diagram of abelian groups, defined on a small category I .

There is a simplicial abelian group $E_I A$, with

$$E_I A_n = \bigoplus_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} A(i_0)$$

and with simplicial structure maps θ^* defined for $\theta : \mathbf{m} \rightarrow \mathbf{n}$ by the commutative diagrams

$$\begin{array}{ccc} A(i_0) & \xrightarrow{\alpha_*} & A(i_{\theta(0)}) \\ \text{in}_\sigma \downarrow & & \downarrow \text{in}_{\theta^*(\sigma)} \\ \bigoplus_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} A(i_0) & \xrightarrow{\theta^*} & \bigoplus_{\gamma: j_0 \rightarrow \dots \rightarrow j_m} A(j_0) \end{array}$$

where $\alpha : i_0 \rightarrow i_{\theta(0)}$ is the morphism of I defined by θ .

The simplicial abelian group $E_I A$ defines the homotopy colimit within simplicial abelian groups.

Specifically, every diagram $B : I \rightarrow s\mathbf{Ab}$ of simplicial abelian groups determines a bisimplicial abelian group $E_I B$ with horizontal objects

$$E_I B_n = \bigoplus_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} B(i_0).$$

There is a **projective model structure** on $s\mathbf{Ab}^I$, for which $f : A \rightarrow B$ is a weak equivalence (respectively fibration) if and only if each map $f : A_i \rightarrow B_i$ is a weak equivalence (respectively fibration) of simplicial abelian groups (exercise).

Lemma 24.1. *The canonical map*

$$E_I B \rightarrow \varinjlim_I B,$$

induces a weak equivalence of simplicial abelian groups

$$\pi : d(E_I B) \rightarrow \varinjlim_I B$$

if B is projective cofibrant.

Proof. The generating projective cofibrations are induced from the generating projective cofibrations

$$j \times 1 : K \times \text{hom}(i, \) \rightarrow L \times \text{hom}(i, \)$$

of I -diagrams of simplicial sets by applying the free abelian group functor.

There is an isomorphism

$$E_I \mathbb{Z}(X) \cong \mathbb{Z}(\underline{\text{holim}}_I X)$$

for all I -diagrams of simplicial sets X .

The map

$$E_I \mathbb{Z}(\text{hom}(i, \) \times K) \rightarrow \varinjlim_I \mathbb{Z}(\text{hom}(i, \) \times K)$$

is the result of applying the free abelian group functor to a diagonal weak equivalence of bisimplicial sets.

Every pointwise weak equivalence of I -diagrams $A \rightarrow B$ induces a diagonal weak equivalence

$$d(E_I A) \rightarrow d(E_I B).$$

This is a consequence of Lemma 24.2. □

Lemma 24.2. *Every level weak equivalence $A \rightarrow B$ of bisimplicial abelian groups induces a weak equivalence $d(A) \rightarrow d(B)$.*

Lemma 24.2 follows from Lemma 21.1 (the bisimplicial sets result).

Corollary 24.3. *Suppose the level weak equivalence $p : A \rightarrow B$ is a projective cofibrant replacement of an I -diagram B .*

Then there are weak equivalences

$$d(E_I B) \xleftarrow{\simeq} d(E_I A) \xrightarrow{\simeq} \varinjlim_I A.$$

Example: Suppose $A : I \rightarrow \mathbf{Ab}$ is a diagram of abelian groups. Write \mathbf{Ab}^I for the category of such I -diagrams and natural transformations.

\mathbf{Ab}^I has a set of projective generators, ie. all functors $\mathbb{Z}(\text{hom}(i, _))$ obtained by applying the free abelian group functor to the functors $\text{hom}(i, _)$, $i \in \text{Ob}(I)$.

It follows that every I -diagram $A : I \rightarrow \mathbf{Ab}$ of abelian groups has a projective resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

The I -diagram $\Gamma(P_*)$ of simplicial abelian groups is projective cofibrant (exercise), so there are weak equivalences of simplicial abelian groups

$$E_I A = d(E_I A) \xleftarrow{\simeq} d(E_I \Gamma(P_*)) \xrightarrow{\simeq} \varinjlim_I \Gamma(P_*) \cong \Gamma(\varinjlim_I P_*).$$

Thus, there are isomorphisms

$$\pi_k(E_I A) \cong \pi_k(\Gamma(\varinjlim_I P_*)) \cong H_k(\varinjlim_I P_*).$$

We have proved the following:

Lemma 24.4. *There are natural isomorphisms*

$$\pi_k(E_I A) \cong L(\varinjlim_I)_k(A)$$

for all I -diagrams of abelian groups A .

In other words, the homotopy (or homology) groups of $E_I A$ coincide with the left derived functors of the colimit functor in abelian groups.

Remark: Exactly the same script works for diagrams of simplicial modules over an arbitrary commutative unitary ring R .

Example: Suppose G is a group, and let $R(G)$ be the corresponding group-algebra over R . An $R(G)$ -module, or simply a G -module in $R - \mathbf{Mod}$, is a diagram

$$M : G \rightarrow R - \mathbf{Mod},$$

and the higher derived functors of \varinjlim_G for M are the group homology groups $H_n(G, M)$, as defined classically.

In effect, one can show that there is an isomorphism of simplicial R -modules

$$\begin{aligned} L(\varinjlim_G)_k(M) &= H_k(E_G M) \\ &\cong H_k(R(EG) \otimes_G M) = H_k(G, M). \end{aligned}$$

Here, $EG = B(* / G)$ is the standard contractible cover of BG so $R(EG) \rightarrow R$ is a free G -resolution of the trivial G -module R .

$R(EG) \otimes_G M$ is the **Borel construction** for the G -module M .

The R -module $\varinjlim_G M$ is the module of **coinvariants** of the G -module M , and it is common to write

$$M/G = \varinjlim_G M.$$

Example: Suppose that $A : \Delta^{op} \rightarrow s\mathbf{Ab}$ is a bisimplicial abelian group.

The colimit $\varinjlim_n A_n$ is the coequalizer

$$A_1 \rightrightarrows A_0 \rightarrow \pi_0 A = \varinjlim_n A_n$$

of the face maps $d_0, d_1 : A_1 \rightarrow A_0$.

The bisimplicial set $\Delta^n \tilde{\times} K$ has (horizontal) colimit

$$\pi_0 \Delta^n \times K \cong K.$$

It follows that the map

$$\mathbf{Z}(\Delta^n \tilde{\times} K) \rightarrow \varinjlim_p \mathbf{Z}(\Delta_p^n \times K)$$

is a levelwise equivalence (in vertical degrees) of bisimplicial abelian groups. This implies that the bisimplicial abelian group map

$$A \rightarrow \pi_0 A = \varinjlim_n A_n$$

is a weak equivalence in all vertical degrees for all projective cofibrant objects A , and therefore in-

duces a diagonal weak equivalence

$$d(A) \xrightarrow{\simeq} \varinjlim_n A_n$$

for all such objects A .

It follows that if $A \rightarrow B$ is a projective cofibrant resolution of a bisimplicial abelian group B , then there are weak equivalences

$$d(B) \xleftarrow{\simeq} d(A) \xrightarrow{\simeq} \varinjlim_n A_n,$$

and so the diagonal $d(B)$ is naturally equivalent to the homotopy colimit of the simplicial object A .

Cohomology

There is a cohomological version of the theory presented so far in this section. A little more technology is involved.

1) The category \mathbf{Ab}^I of I -diagrams of abelian groups has enough injectives.

2) If A is an I -diagram of abelian groups, then there is an isomorphism of cochain complexes

$$\mathrm{hom}(B(I/?), A) \cong \prod^* A.$$

3) The functor $\mathrm{hom}(_, J)$ is exact if J is injective (exercise), and thus takes weak equivalences $X \rightarrow$

Y of I -diagrams of simplicial sets to cohomology isomorphisms $\text{hom}(Y, J) \rightarrow \text{hom}(X, J)$.

The canonical map $B(I/?) \rightarrow *$ is a weak equivalence of I -diagrams, so the morphism

$$\text{hom}(*, J) \rightarrow \text{hom}(B(I/?), J)$$

is a cohomology isomorphism if J is injective. Thus, there are isomorphisms

$$H^k \prod^* J \cong \begin{cases} \varprojlim_I J & \text{if } k = 0, \text{ and} \\ 0 & \text{if } k > 0. \end{cases}$$

4) More generally, there are isomorphisms

$$H^k \prod^* A \cong R(\varprojlim_I)^k A =: \varprojlim_I^k A$$

for $k \geq 0$ and for all I -diagrams A .

In effect, A has an injective resolution $A \rightarrow J^*$ and both (cohomological) spectral sequences for the bicomplex $\prod^* J^*$ collapse.

5) If A is an I -diagram of abelian groups, then there is an isomorphism

$$[* , K(A, n)] \cong \varprojlim_I^n (A), \quad (1)$$

where $[,]$ denotes morphisms in the homotopy category of I -diagrams of simplicial sets.

The best argument that I know of for the isomorphism (1) appears in [4] (also [5]).

The theory of higher right derived functors of inverse limit is a type of sheaf cohomology theory.

6) There are isomorphisms

$$\begin{aligned}
 \pi_0 \underset{\leftarrow}{\text{holim}}_I K(A, n) &\cong \pi_0 \underset{\leftarrow}{\text{holim}}_I Z \\
 &\cong \pi_0 \underset{\leftarrow}{\text{lim}}_I Z \\
 &\cong [* , Z] \\
 &\cong [* , K(A, n)] \\
 &\cong \underset{\leftarrow}{\text{lim}}_I^n A,
 \end{aligned}$$

where $K(A, n) \rightarrow Z$ is an injective fibrant model of $K(A, n)$.

The object $K(A, n)$ is a de-looping of $K(A, n - 1)$, so there are isomorphisms

$$\pi_k \underset{\leftarrow}{\text{holim}}_I K(A, n) \cong \begin{cases} \underset{\leftarrow}{\text{lim}}_I^{n-k} A & \text{if } 0 \leq k \leq n, \text{ and} \\ 0 & \text{if } k > n \end{cases}$$

25 Spectral sequences for a bicomplex

This section contains a very basic introduction to spectral sequences.

We shall only explicitly discuss the spectral sequences in homology which are associated to a bicomplex.

These spectral sequences, their cohomological analogs (used at the end of Section 24), and the Bousfield-Kan spectral sequence for a tower of fibrations [1], are the most common prototypes for spectral sequences that one meets in nature.

Most of the material of this section in Mac Lane's "Homology" [6]. There are many other sources.

A **bicomplex** C consists of an array of abelian groups $C_{p,q}$, $p, q \geq 0$ and morphisms

$$\partial_v : C_{p,q} \rightarrow C_{p,q-1} \quad \text{and} \quad \partial_h : C_{p,q} \rightarrow C_{p-1,q},$$

such that

$$\begin{aligned} \partial_v^2 = \partial_h^2 = 0 \quad \text{and} \\ \partial_v \partial_h + \partial_h \partial_v = 0. \end{aligned}$$

A **morphism** $f : C \rightarrow D$ of bicomplexes consists of morphisms $f : C_{p,q} \rightarrow D_{p,q}$ that respect the differentials.

Write Ch_+^2 for the corresponding category.

There is a functor

$$\text{Tot} : Ch_+^2 \rightarrow Ch_+$$

taking values in ordinary chain complexes with

$$\text{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

and with differential $\partial : \text{Tot}(C)_n \rightarrow \text{Tot}(C)_{n-1}$ defined on the summand $C_{p,q}$ by

$$\partial(x) = \partial_v(x) + \partial_h(x).$$

Every bicomplex C has two filtrations, horizontal and vertical.

The p^{th} stage $F_p C$ of the **horizontal filtration** has

$$F_p C_{r,s} = \begin{cases} C_{r,s} & \text{if } r \leq p, \text{ and} \\ 0 & \text{if } r > p. \end{cases}$$

Then

$$0 = F_{-1}C \subset F_0C \subset F_1C \subset \dots$$

and

$$\bigcup_p F_p(C) = C$$

The functor $C \mapsto \text{Tot}(C)$ is exact, so this filtration on C induces a filtration on $\text{Tot}(C)$.

One filters $\text{Tot}(C)_n$ in finitely many stages:

$$0 = F_1 \text{Tot}(C)_n \subset F_0 \text{Tot}(C)_n \subset \dots \subset F_n \text{Tot}(C)_n = \text{Tot}(C)_n.$$

Generally, the long exact sequences in homology associated to the exact sequences

$$0 \rightarrow F_{p-1}C \xrightarrow{i} F_p C \xrightarrow{p} F_p C / F_{p-1}C \rightarrow 0$$

arising from a filtration $\{F_p C\}$ on a chain complex C fit together to define a **spectral sequence** for the filtered complex.

This spectral sequence arises from the “ladder diagram”

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \\
 \downarrow & & \downarrow & & \\
 H_{p+q}(F_{p-2}) & & H_{p+q-1}(F_{p-3}) & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 H_{p+q}(F_{p-1}) & & H_{p+q-1}(F_{p-2}) \longrightarrow H_{p+q-1}(F_{p-2}/F_{p-3}) & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 H_{p+q}(F_p) \xrightarrow{p_*} H_{p+q}(F_p/F_{p-1}) \xrightarrow{\partial} H_{p+q-1}(F_{p-1}) & & \downarrow i_* & & \\
 \downarrow i_* & & H_{p+q-1}(F_p) \xrightarrow{p_*} H_{p+q-1}(F_p/F_{p-1}) & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 \vdots & & \vdots & & \\
 \downarrow & & \downarrow & & \\
 H_{p+q}(C) & & H_{p+q-1}(C) & &
 \end{array}$$

Set

$$Z_r^{p,q} = \{x \in H_{p+q}(F_p/F_{p-1}) \mid \partial(x) \in \text{im}(i_*^{r-1})\}$$

and

$$B_r^{p,q} = p_*(\ker(i_*^{r-1})),$$

and then define

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}}.$$

for $r \geq 1$. Here, we adopt the convention that $i_*^0 = 1$, so that

$$E_1^{p,q} = H_{p+q}(F_p/F_{p-1})$$

This is cheating slightly (this only works for bi-complexes), but set

$$E_\infty^{p,q} = \frac{\ker(\partial)}{p_*(\ker(H_{p+q}(F_p) \rightarrow H_{p+q}(C)))}.$$

Finally, define

$$F_p H_{p+q}(C) = \text{im}(H_{p+q}(F_p) \rightarrow H_{p+q}(C)).$$

Given $[x] \in E_r^{p,q}$ represented by $x \in Z_r^{p,q}$ choose $y \in H_{p+q}(F_{p-r})$ such that $i_*^{r-1}(y) = \partial(x)$. Then the assignment $[x] \mapsto [p_*(y)]$ defines a homomorphism

$$d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1},$$

and this homomorphism is natural in filtered complexes.

Then we have the following:

Lemma 25.1. 1) We have the relation $d_r^2 = 0$, and there is an isomorphism

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r : E_r^{p,q} \rightarrow E_r^{p-r,q+r-1})}{\operatorname{im}(d_r : E_r^{p+r,q-r+1} \rightarrow E_r^{p,q})}.$$

2) There are isomorphisms

$$E_r^{p,q} \cong E_\infty^{p,q}$$

for $r > p, q + 2$.

3) There are short exact sequences

$$0 \rightarrow F_{p-1}H_{p+q}(C) \rightarrow F_pH_{p+q}(C) \rightarrow E_\infty^{p,q} \rightarrow 0.$$

The proof is an exercise — chase some elements.

In general (ie. for general filtered complexes),

$$E_1^{p,q} = H_{p+q}(F_p/F_{p-1}),$$

and $E_2^{p,q}$ is the homology of the complex with differentials $E_1^{p,q} \rightarrow E_1^{p-1,q}$ given by the composites

$$H_{p+q}(F_p/F_{p-1}) \xrightarrow{\partial} H_{p+q-1}(F_{p-1}) \xrightarrow{p_*} H_{p+q-1}(F_{p-1}/F_{p-2})$$

In the case of the horizontal filtration $F_p \operatorname{Tot}(C)$ for a bicomplex C , there is a natural isomorphism

$$F_p \operatorname{Tot}(C) / F_{p-1} \operatorname{Tot}(C) \cong C_{p,*}[p],$$

so there is an isomorphism

$$E_1^{p,q} \cong H_q(C_{p,*})$$

The differential d_1 is the homomorphism

$$H_q(C_{p,*}) \xrightarrow{\partial_{h*}} H_q(C_{p-1,*})$$

which is induced by the horizontal differential.

It follows, that for the horizontal filtration on the total complex $\text{Tot}(C)$ of a bicomplex C , there is a spectral sequence with

$$E_2^{p,q} = H_p^h(H_q^v C) \Rightarrow H_{p+q}(\text{Tot}(C)).$$

In particular, the spectral sequence converges to $H_*(\text{Tot}(C))$ in the sense that the filtration quotients $E_\infty^{p,q}$ determine $H_*(\text{Tot}(C))$.

Here's an example of how it all works:

Lemma 25.2. *Suppose $f : C \rightarrow D$ is a morphism of bicomplexes such that for some $r \geq 1$ the induced morphisms $E_r^{p,q}(C) \rightarrow E_r^{p,q}(D)$ are isomorphisms for all $p, q \geq 0$.*

Then the induced map $\text{Tot}(C) \rightarrow \text{Tot}(D)$ is a homology isomorphism.

Lemma 25.2 is sometimes called the *Zeeman comparison theorem*.

Proof. The map f induces isomorphisms

$$E_s^{p,q}(C) \xrightarrow{\cong} E_s^{p,q}(D)$$

for all $s \geq r$ (because all such E_s -terms are computed by taking homology groups, inductively in $s \geq r + 1$). It follows that all induced maps

$$E_\infty^{p,q}(C) \rightarrow E_\infty^{p,q}(D)$$

are isomorphisms. But then, starting with the morphism

$$\begin{array}{ccc} E_\infty^{0,p+q}(C) & \xrightarrow{\cong} & E_\infty^{0,p+q}(D) \\ \cong \downarrow & & \downarrow \cong \\ F_0 H_{p+q}(\text{Tot}(C)) & \longrightarrow & F_0 H_{p+q}(\text{Tot}(D)) \end{array}$$

and using the fact that the induced maps on successive filtration quotients are the isomorphisms

$$E_\infty^{r,p+q-r}(C) \xrightarrow{\cong} E_\infty^{r,p+q-r}(D),$$

one shows inductively that all maps

$$F_r H_{p+q}(\text{Tot}(C)) \rightarrow F_r H_{p+q}(\text{Tot}(D))$$

are isomorphisms, including the case $r = p + q$ which is the map

$$H_{p+q}(\text{Tot}(C)) \rightarrow H_{p+q}(\text{Tot}(D)).$$

This is true for all total degrees $p + q$, so the map $\text{Tot}(C) \rightarrow \text{Tot}(D)$ is a quasi-isomorphism. \square

Example: Suppose A is a bisimplicial abelian group. Then the Generalized Eilenberg-Zilber Theorem (Theorem 26.1) asserts that there is a natural chain homotopy equivalence of chain complexes

$$d(A) \simeq \text{Tot}(A)$$

where $\text{Tot}(A)$ is the total complex of the associated (Moore) bicomplex. Filtering A in the horizontal direction therefore gives a spectral sequence with

$$E_2^{p,q} = \pi_p^h(\pi_q^v(A)) \Rightarrow \pi_{p+q}d(A). \quad (2)$$

This spectral sequence is natural in bisimplicial abelian groups A .

This spectral sequence can be used to give an alternate proof of Lemma 24.4. If $A \rightarrow B$ is a level equivalence of bisimplicial abelian groups, then there is an E_1 -level isomorphism

$$\pi_q(A_{p,*}) \xrightarrow{\cong} \pi_q(B_{p,*})$$

for all $p, q \geq 0$.

Use Lemma 25.2 for the spectral sequence (2) to show that the map $d(A) \rightarrow d(B)$ is a weak equivalence.

Application: The Lyndon-Hochschild-Serre spectral sequence

Suppose $f : C \rightarrow D$ is a functor between small categories, and recall the bisimplicial set map

$$\bigsqcup_{d_0 \rightarrow \cdots \rightarrow d_n} B(f/d_0) \rightarrow BC$$

of Section 23. Lemma 23.3 says that this map is a diagonal weak equivalence.

The free abelian group functor preserves diagonal weak equivalences, so there is a spectral sequence

$$E_2^{p,q} = L(\varinjlim)_p H_q(B(f/?), \mathbb{Z}) \Rightarrow H_{p+q}(BC, \mathbb{Z}). \quad (3)$$

The derived colimit functors are computed over the base category D .

In the special case where f is a surjective group homomorphism $G \rightarrow H$ with kernel K , this is a form of the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H_p(H, H_q(BK, \mathbb{Z})) \Rightarrow H_{p+q}(BG, \mathbb{Z}). \quad (4)$$

We can put in other coefficients if we want.

To see that the E_2 -term of (4) has the indicated form, take a set-theoretic section $\sigma : H \rightarrow G$ of the group homomorphism f such that $\sigma(e) = e$.

Conjugation $x \mapsto \sigma(h)x\sigma(h)^{-1}$, defines a group isomorphism $c_{\sigma(h)} : K \rightarrow K$ and hence an isomorphism

$$h_* : H_*(BK, \mathbb{Z}) \rightarrow H_*(BK, \mathbb{Z})$$

in homology. The map h_* is independent of the choice of section σ because any two pre-images of h determine homotopic maps $K \rightarrow K$ (ie. the two isomorphisms differ by conjugation by an element of K).

This action of H on $H_*(BK, \mathbb{Z})$ is the one appearing in the description of the E_2 -term of (4).

The objects of $f/*$ are the elements of H , and a morphism $g : h \rightarrow h'$ in $f/*$ is an element $g \in G$ such that $h'f(g) = h$.

There is a functor $K \rightarrow f/*$ defined by sending $k \in K$ to the morphism $k : e \rightarrow e$.

There is a functor $f/* \rightarrow K$ which is defined by sending the morphism $g : h \rightarrow h'$ to the element $\sigma(h')g\sigma(h)^{-1}$.

$\sigma(e) = e$, so the composite functor

$$K \rightarrow f/* \rightarrow K$$

is the identity, while the elements $\sigma(h)$, $h \in H$ de-

find a homotopy from the composite

$$f/* \rightarrow K \rightarrow f/*$$

to the identity on $f/*$.

Finally, composition with $\alpha \in H$ defines the functor $\alpha_* : f/* \rightarrow f/*$ in the description of the bisimplicial set for $f : G \rightarrow H$, and there is a homotopy commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & f/* \\ c_{\sigma(\alpha)} \downarrow & & \downarrow \alpha_* \\ K & \longrightarrow & f/* \end{array}$$

Thus, the action of α on $H_*(B(f/*), \mathbb{Z})$ coincides with the morphism $\alpha_* : H_*(BK, \mathbb{Z}) \rightarrow H_*(BK, \mathbb{Z})$ displayed above, up to isomorphism.

Example: Consider the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

All three groups are abelian, so all conjugation actions are trivial and there is a spectral sequence with

$$E_2^{p,q} = H_p(B(\mathbb{Q}/\mathbb{Z}), H_q(B\mathbb{Z}, \mathbb{Q})) \Rightarrow H_{p+q}(B\mathbb{Q}, \mathbb{Q}).$$

$S^1 \simeq B\mathbb{Z}$, so there are isomorphisms

$$H_q(B\mathbb{Z}, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } q = 0, 1, \text{ and} \\ 0 & \text{if } q > 1. \end{cases}$$

The group \mathbb{Q}/\mathbb{Z} is all torsion, so that

$$H_q(B(\mathbb{Q}/\mathbb{Z}), \mathbb{Q}) = 0$$

for $q \geq 1$.

The E_2 -term for the spectral sequence therefore collapses, so the “edge homomorphism”

$$H_*(B\mathbb{Z}, \mathbb{Q}) \rightarrow H_*(B\mathbb{Q}, \mathbb{Q})$$

is an isomorphism.

To see the claim about torsion groups, observe that torsion abelian groups are filtered colimits of finitely generated torsion abelian groups, and a finitely generated torsion abelian group is a finite direct sum of cyclic groups.

It therefore suffices, by a Künneth formula argument (see (11) below) to show that

$$H_p(\mathbb{Z}/n, \mathbb{Q}) = H_p(B(\mathbb{Z}/n), \mathbb{Q}) = 0$$

for $p > 0$.

The abelian group \mathbb{Z} , as a trivial \mathbb{Z}/n -module, has a free resolution by \mathbb{Z}/n -modules

$$\dots \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/n) \xrightarrow{1-t} \mathbb{Z}(\mathbb{Z}/n) \rightarrow \mathbb{Z} \rightarrow 0 \quad (5)$$

where $1 - t$ is multiplication by group-ring element $1 - t$ and t is the generator of the group \mathbb{Z}/n .

The map N is multiplication by the “norm element”

$$N = 1 + t + t^2 + \cdots + t^{n-1}.$$

Tensoring the resolution with the trivial \mathbb{Z}/n -module \mathbb{Z} gives the chain complex

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

and it follows that

$$H_p(B\mathbb{Z}/n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{if } p = 2n, n > 0, \text{ and} \\ \mathbb{Z}/n & \text{if } p = 2n + 1, n \geq 0. \end{cases} \quad (6)$$

Tensoring with \mathbb{Q} (which is exact) therefore shows that $H_p(B\mathbb{Z}/n, \mathbb{Q}) = 0$ for $p > 0$.

We could equally well tensor the resolution (5) with the trivial \mathbb{Z}/n -module \mathbb{Q} and get the same answer, because \mathbb{Q} is uniquely n -divisible.

26 The Eilenberg-Zilber Theorem

Every bisimplicial abelian group A has a naturally associated bicomplex $M(A)$ with

$$M(A)_{m,n} = A_{m,n},$$

and with horizontal boundaries

$$\partial_h = \sum_{i=0}^m (-1)^i d_i : A_{m,n} \rightarrow A_{m-1,n}$$

and vertical boundaries

$$\partial_v = \sum_{i=0}^n (-1)^{m+i} d_i : A_{m,n} \rightarrow A_{m,n-1}.$$

One checks that

$$\partial_h \partial_v + \partial_v \partial_h = 0$$

in all bidegrees — the signs were put in to achieve this formula.

Here is the Generalized Eilenberg-Zilber Theorem of Dold-Puppe [2], [3, IV.2.2]:

Theorem 26.1 (Dold-Puppe). *Suppose that A is a bisimplicial abelian group.*

Then the chain complexes $d(A)$ and $\text{Tot}(A)$ are naturally chain homotopy equivalent.

Proof. The standard Eilenberg-Zilber Theorem says that there are natural chain maps

$$f : \mathbb{Z}(K \times L) \rightarrow \text{Tot}(\mathbb{Z}(K) \otimes \mathbb{Z}(L))$$

(Moore complexes) and

$$g : \text{Tot}(\mathbb{Z}(K) \otimes \mathbb{Z}(L)) \rightarrow \mathbb{Z}(K \times L),$$

and there are natural chain homotopies $fg \simeq 1$ and $gf \simeq 1$ for simplicial sets K and L .

The Eilenberg-Zilber Theorem specializes to (is equivalent to — exercise) the existence of chain maps

$$d(\mathbb{Z}(\Delta^{p,q})) = \mathbb{Z}(\Delta^p \times \Delta^q) \xrightarrow{f} \text{Tot}(\mathbb{Z}(\Delta^p) \otimes \mathbb{Z}(\Delta^q)) = \text{Tot}(\mathbb{Z}(\Delta^{p,q}))$$

and

$$g : \text{Tot}(\mathbb{Z}(\Delta^{p,q})) \rightarrow d(\mathbb{Z}(\Delta^{p,q}))$$

and chain homotopies $fg \simeq 1$ and $gf \simeq 1$ which are natural in bisimplices $\Delta^{p,q}$.

Every bisimplicial abelian group A is a natural colimit of the diagrams

$$\begin{array}{ccc} A_{p,q} \otimes \mathbb{Z}(\Delta^{r,s}) & \xrightarrow{1 \otimes (\gamma, \theta)} & A_{p,q} \otimes \mathbb{Z}(\Delta^{p,q}) \\ (\gamma, \theta)^* \otimes 1 \downarrow & & \\ A_{r,s} \otimes \mathbb{Z}(\Delta^{r,s}) & & \end{array}$$

where $(\gamma, \theta) : (\mathbf{r}, \mathbf{s}) \rightarrow (\mathbf{p}, \mathbf{q})$ varies over the morphisms of $\Delta \times \Delta$, and the maps

$$\gamma_{p,q} : A_{p,q} \otimes \mathbb{Z}(\Delta^{p,q}) \rightarrow A$$

given by $(a, (\gamma, \theta)) \mapsto (\gamma, \theta)^*(a)$ define the colimit.

There are isomorphisms

$$\begin{aligned} d(B \otimes A) &\cong B \otimes d(A), \\ \text{Tot}(B \otimes A) &\cong B \otimes \text{Tot}(A) \end{aligned}$$

for bisimplicial abelian groups A and abelian groups B , and these isomorphisms are natural in both A and B . The functors Tot and d are also right exact.

It follows that f and g induce natural chain maps

$$f_* : d(A) \rightarrow \text{Tot}(A), \quad g_* : \text{Tot}(A) \rightarrow d(A)$$

for all simplicial abelian groups A .

The chain homotopies $f_*g_* \simeq 1$ and $g_*f_* \simeq 1$ induce natural chain homotopies

$$\begin{aligned} f_*g_* &\simeq 1 : \text{Tot}(A) \rightarrow \text{Tot}(A), \\ g_*f_* &\simeq 1 : d(A) \rightarrow d(A) \end{aligned}$$

for all bisimplicial abelian groups A . □

Remarks: 1) The proof of Theorem 26.1 which appears in [3, p.205] contains an error: the sequence

$$\bigoplus_{\tau \rightarrow \sigma} \mathbb{Z}(\Delta^{r,s}) \rightarrow \bigoplus_{\Delta^{p,q} \rightarrow A} \mathbb{Z}(\Delta^{p,q}) \rightarrow A \rightarrow 0$$

is not exact, which means that A is not a colimit of its bisimplices in general. The problem is fixed

by using the co-end description of A that you see above.

2) The maps f and g have classical explicit models, namely the Alexander-Whitney map and shuffle map, respectively. See [6, VIII.8] for a full discussion.

Recall that $M(A)$ denotes the Moore chain complex of a simplicial abelian group A .

The Alexander-Whitney map

$$f : M(A \otimes B) \rightarrow \text{Tot}(M(A) \otimes M(B)) \quad (7)$$

is defined, for simplicial abelian groups A and B by

$$f(a \otimes b) = \sum_{0 \leq p \leq n} a|_{[0, \dots, p]} \otimes b|_{[p, \dots, n]}.$$

Here, $a \in A_n$ and $b \in B_n$ are n -simplices. The “front p -face” $a|_{[0, \dots, p]}$ is defined by

$$\Delta^p \xrightarrow{[0, \dots, p]} \Delta^n \xrightarrow{a} A.$$

The “back $(n - p)$ -face” $b|_{[p, \dots, n]}$ is defined by

$$\Delta^{n-p} \xrightarrow{[p, \dots, n]} \Delta^n \xrightarrow{b} B.$$

The Eilenberg-Zilber Theorem follows from

Lemma 26.2. 1) *The object*

$$(\mathbf{p}, \mathbf{q}) \mapsto \mathbb{Z}(\Delta^p \times \Delta^q)$$

is a projective cofibrant $(\Delta \times \Delta)$ -diagram of simplicial abelian groups.

2) *The object*

$$(\mathbf{p}, \mathbf{q}) \mapsto \text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q))$$

is a projective cofibrant $(\Delta \times \Delta)$ -diagram of chain complexes.

To see that Lemma 26.2 implies the Eilenberg-Zilber Theorem, observe that there is a natural chain homotopy equivalence

$$\text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)) \simeq \text{Tot}(M\mathbb{Z}(\Delta^p) \otimes M\mathbb{Z}(\Delta^q))$$

of bicosimplicial chain complexes which is induced by the natural chain homotopy equivalence of Theorem 15.4 (Lecture 06) between normalized and Moore chain complexes.

There is a similar natural chain homotopy equivalence

$$M\mathbb{Z}(\Delta^p \times \Delta^q) \simeq N\mathbb{Z}(\Delta^p \times \Delta^q).$$

Finally, there is a natural chain homotopy equivalence

$$N\mathbb{Z}(\Delta^p \times \Delta^q) \simeq \text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)),$$

since both objects are projective cofibrant resolutions of the constant diagram of chain complexes $\mathbb{Z}(0)$ on $\Delta \times \Delta$ by Lemma 26.2.

We use the following result to prove Lemma 26.2:

Lemma 26.3. *Suppose $p : A \rightarrow B$ is a trivial projective fibration of cosimplicial simplicial abelian groups. Then all induced maps*

$$(p, s) : A^{n+1} \rightarrow B^{n+1} \times_{M^n B} M^n A$$

are trivial fibrations of simplicial abelian groups.

Proof. The map $s : B^{n+1} \rightarrow M^n B$ is surjective for all cosimplicial abelian groups B . In effect, if $x = (0, \dots, 0, x_i, \dots, x_n) \in M^n B$ then

$$s(d^{i+1}x_i) = (0, \dots, 0, x_i, y_{i+1}, \dots, y_n),$$

and $x - s(d^{i+1}x_i)$ is of the form

$$x - s(d^{i+1}x_i) = (0, \dots, 0, z_{i+1}, \dots, z_n) =: z.$$

Thus, inductively, if $z = s(v)$ for some $v \in A^{n+1}$ then $x = s(d^{i+1}x_i + v)$.

Write

$$M_{(0,i)}^n A = \{(x_0, \dots, x_i) \mid x_i \in A^n, s^i x_j = s^{j-1} x_i \text{ for } i < j\}.$$

Then $M^n A = M_{(0,n)}^n A$, and there are pullback dia-

grams

$$\begin{array}{ccc}
 M_{(0,i+1)}^n & \longrightarrow & A^n \\
 \downarrow & & \downarrow s \\
 M_{(0,i)}^n & \xrightarrow{s^i} & M_{(0,i)}^{n-1}A
 \end{array}$$

in which the two unnamed arrows are projections.

Suppose K is a cosimplicial object in $s\mathbf{Ab}$ such that all objects K^n are acyclic.

Then under the inductive assumption that $s : K^n \rightarrow M_{(0,i)}^{n-1}K$ is a trivial fibration we see that the projection $M_{(0,i+1)}^n K \rightarrow K^n$ is a trivial fibration, and so the map $s : K^{n+1} \rightarrow M_{(0,i+1)}^n K$ is a weak equivalence.

This is true for all $i < n$, and it follows that the map $s : K^{n+1} \rightarrow M^n K$ is a trivial fibration.

If K is the kernel of the projective trivial fibration $p : A \rightarrow B$, then there is an induced comparison of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K^{n+1} & \longrightarrow & A^{n+1} & \longrightarrow & B^{n+1} \longrightarrow 0 \\
 & & \downarrow s & & \downarrow (p,s) & & \downarrow 1 \\
 0 & \longrightarrow & M^n K & \longrightarrow & B^{n+1} \times_{M^n B} M^n A & \longrightarrow & B^{n+1} \longrightarrow 0
 \end{array}$$

so the map (p, s) is a weak equivalence. \square

Corollary 26.4. *The cosimplicial simplicial abelian group $\mathbf{n} \mapsto \mathbb{Z}(\Delta^n)$ is projective cofibrant.*

Proof. Suppose $p : A \rightarrow B$ is a projective trivial fibration. Solving a lifting problem

$$\begin{array}{ccc} & & A \\ & \nearrow \text{dotted} & \downarrow p \\ \mathbb{Z}(\Delta) & \longrightarrow & B \end{array}$$

amounts to inductively solving lifting problems

$$\begin{array}{ccc} \partial\Delta^{n+1} & \longrightarrow & A^{n+1} \\ \downarrow & \nearrow \text{dotted} & \downarrow (p,s) \\ \Delta^{n+1} & \longrightarrow & B^{n+1} \times_{M^n B} M^n A \end{array}$$

and this can be done by the previous Lemma. \square

Proof of Lemma 26.2. Suppose $q : C \rightarrow D$ is a projective trivial fibration of $(\Delta \times \Delta)$ -diagrams of simplicial abelian groups. Then all maps

$$(q, s) : C^{n+1} \rightarrow D^{n+1} \times_{M^n D} M^n C$$

are projective trivial fibrations of cosimplicial simplicial abelian groups, by Lemma 26.3.

Write $\Delta \times \Delta$ for the bicosimplicial diagram

$$(\mathbf{p}, \mathbf{q}) \mapsto \Delta^{\mathbf{p}} \times \Delta^{\mathbf{q}}$$

of simplicial sets.

Then lifting problems

$$\begin{array}{ccc} & & C \\ & \nearrow & \downarrow q \\ \Delta \times \Delta & \longrightarrow & D \end{array}$$

can be solved by inductively solving the lifting problems

$$\begin{array}{ccc} \mathbb{Z}(\Delta \times \partial\Delta^{n+1}) & \longrightarrow & C^{n+1} \\ \downarrow & \nearrow & \downarrow (q,s) \\ \mathbb{Z}(\Delta \times \Delta^{n+1}) & \longrightarrow & D^{n+1} \times_{M^n D} M^n C \end{array}$$

in cosimplicial simplicial abelian groups.

For that, it suffices to show that the map

$$\mathbb{Z}(\Delta \times \partial\Delta^{n+1}) \rightarrow \mathbb{Z}(\Delta \times \Delta^{n+1})$$

is a projective cofibration, but this follows from the observation that the maps

$$\mathbb{Z}(\partial\Delta^m \times \Delta^{n+1}) \cup \mathbb{Z}(\Delta^m \times \partial\Delta^{m+1}) \rightarrow \mathbb{Z}(\Delta^m \times \Delta^{n+1})$$

are cofibrations of simplicial abelian groups xfor $m \geq 0$, with Lemma 26.3.

We have proved statement 1) of Lemma 26.2.

The second statement of Lemma 26.2 has a very similar proof. If $q : C \rightarrow D$ is a projective trivial fibration of bicosimplicial chain complexes, then

all maps

$$(p, s) : C^{n+1} \rightarrow D^{n+1} \times_{M^n D} M^n C$$

are projective trivial fibrations of cosimplicial chain complexes, by Lemma 26.3. Write

$$\text{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta))$$

for the bicosimplicial chain complex

$$(\mathbf{p}, \mathbf{q}) \mapsto \text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)).$$

Then solving lifting problems

$$\begin{array}{ccc} & & \mathbf{C} \\ & \nearrow & \downarrow q \\ \text{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta)) & \longrightarrow & \mathbf{D} \end{array}$$

amounts to inductively solving lifting problems

$$\begin{array}{ccc} \text{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\partial\Delta^{n+1})) & \longrightarrow & \mathbf{C}^{n+1} \\ \downarrow i & \nearrow & \downarrow (q,s) \\ \text{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta^{n+1})) & \longrightarrow & \mathbf{D}^{n+1} \times_{M^n D} M^n C \end{array}$$

For this, we show that all maps i are projective cofibrations of cosimplicial chain complexes, but this reduces to showing that each of the maps

$$\begin{aligned} & \text{Tot}(N\mathbb{Z}(\Delta^m) \otimes N\mathbb{Z}(\partial\Delta^{n+1})) \cup \text{Tot}(N\mathbb{Z}(\partial\Delta^m) \otimes N\mathbb{Z}(\Delta^{n+1})) \\ & \rightarrow \text{Tot}(N\mathbb{Z}(\Delta^m) \otimes N\mathbb{Z}(\Delta^{n+1})) \end{aligned}$$

are cofibrations of chain complexes.

This last morphism is defined by freely adjoining the chain $\iota_m \otimes \iota_n$, so it is a cofibration. \square

Remark: The proof of the Eilenberg-Zilber Theorem that one finds in old textbooks uses the method of acyclic models.

27 Universal coefficients, Künneth formula

Suppose X is a simplicial set, and that A is an abelian group.

Recall that the n^{th} homology group $H_n(X, A)$ of X with coefficients in A is defined by

$$H_n(X, A) = H_n(\mathbb{Z}(X) \otimes_{\mathbb{Z}} A),$$

where $\mathbb{Z}(X)$ denotes both a free simplicial abelian group and its associated Moore complex.

The ring \mathbb{Z} is a principal ideal domain, so A (a \mathbb{Z} -module) has a free resolution

$$0 \rightarrow F_2 \xrightarrow{i} F_1 \xrightarrow{p} A \rightarrow 0.$$

All abelian groups $\mathbb{Z}(X_n)$ are free, and tensoring with a free abelian group is exact, so there is a short exact sequence of chain complexes

$$0 \rightarrow \mathbb{Z}(X) \otimes F_2 \xrightarrow{1 \otimes i} \mathbb{Z}(X) \otimes F_1 \xrightarrow{1 \otimes p} \mathbb{Z}(X) \otimes A \rightarrow 0.$$

The long exact sequence in H_* has the form

$$\dots \xrightarrow{\partial} H_n(X, F_2) \xrightarrow{(1 \otimes i)_*} H_n(X, F_1) \xrightarrow{(1 \otimes p)_*} H_n(X, A) \xrightarrow{\partial} \dots$$

There are commutative diagrams

$$\begin{array}{ccc} H_n(X, F_2) & \xrightarrow{(1 \otimes i)_*} & H_n(X, F_1) \\ \cong \downarrow & & \downarrow \cong \\ H_n(X, \mathbb{Z}) \otimes F_2 & \xrightarrow{1 \otimes i} & H_n(X, \mathbb{Z}) \otimes F_1 \end{array}$$

It follows that there are short exact sequences

$$0 \rightarrow H_n(X, \mathbb{Z}) \otimes A \rightarrow H_n(X, A) \rightarrow \text{Tor}(H_{n-1}(X, \mathbb{Z}), A) \rightarrow 0. \quad (8)$$

These are the **universal coefficients** exact sequences.

Both Z_n (n -cycles) and B_n (n -boundaries) are free abelian groups, and so there is a map $\phi_n : B_n \rightarrow \mathbb{Z}(X)_{n+1}$ such that the diagram of abelian group homomorphisms

$$\begin{array}{ccc} B_n & \xrightarrow{\phi_n} & \mathbb{Z}(X)_{n+1} \\ j \downarrow & & \downarrow \\ Z_n & \xrightarrow{i} & \mathbb{Z}(X)_n \end{array} \quad (9)$$

commutes, where i and j are canonical inclusions.

Write \tilde{Z}_n for the chain complex which is concentrated in degrees n and $n + 1$ and with boundary morphism given by the inclusion j .

The diagram (9) defines a chain map

$$\phi_n : \tilde{Z}_n \rightarrow \mathbb{Z}(X),$$

which induces an isomorphism

$$H_n(\tilde{Z}_n) \cong H_n(\mathbb{Z}(X)),$$

while $H_k(\tilde{Z}_n) = 0$ for $k \neq n$.

Adding up the maps ϕ_n therefore determines a (non-natural) weak equivalence

$$\phi : \bigoplus_{n \geq 0} \tilde{Z}_n \rightarrow \mathbb{Z}(X).$$

The two complexes are cofibrant, so ϕ is a chain homotopy equivalence and in particular there is a chain homotopy inverse

$$\psi : \mathbb{Z}(X) \rightarrow \bigoplus_{n \geq 0} \tilde{Z}_n.$$

The map ψ and projection onto the complex \tilde{Z}_n therefore determine a chain map

$$\mathbb{Z}(X) \otimes A \rightarrow \tilde{Z}_n \otimes A$$

Comparing universal coefficients sequences gives a commutative diagram

$$\begin{array}{ccc} H_n(X) \otimes A & \longrightarrow & H_n(X, A) \\ \cong \downarrow & & \downarrow \\ H_n(X) \otimes A & \xrightarrow{\cong} & H_n(\tilde{Z}_n \otimes A) \end{array}$$

It follows that the natural map

$$H_n(X) \otimes A \rightarrow H_n(X, A)$$

from the universal coefficients sequence (8) is non-naturally split.

We have proved

Theorem 27.1 (Universal Coefficients Theorem). *Suppose X is a simplicial set and A is an abelian group.*

There is a short exact sequence

$$\begin{aligned} 0 \rightarrow H_n(X, \mathbb{Z}) \otimes A &\rightarrow H_n(X, A) \\ &\rightarrow \text{Tor}(H_{n-1}(X, \mathbb{Z}), A) \rightarrow 0. \end{aligned}$$

for each $n \geq 1$. This sequence is natural in X , and has a non-natural splitting.

Here's a different take on universal coefficients:

The chain complex $\tilde{Z}_n \otimes A$ has homology

$$H_k(\tilde{Z}_n \otimes A) \cong \begin{cases} \operatorname{Tor}(H_n(X), A) & \text{if } k = n + 1, \\ H_n(X) \otimes A & \text{if } k = n, \\ 0 & \text{if } k \neq n, n + 1, \end{cases}$$

and the chain homotopy equivalence ϕ induces isomorphisms

$$\begin{aligned} H_n(X, A) &\cong H_n\left(\bigoplus_{n \geq 0} \tilde{Z}_n \otimes A\right) \\ &\cong (H_n(X) \otimes A) \oplus \operatorname{Tor}(H_{n-1}(X), A). \end{aligned}$$

Remark: The simplicial set underlying a simplicial abelian group has the homotopy type (non-naturally) of a product of Eilenberg-Mac Lane spaces — see [3, III.2.20].

Suppose C is a chain complex.

Then the chain homotopy equivalence ϕ induces a homology isomorphism

$$\operatorname{Tot}\left(\bigoplus_{n \geq 0} \tilde{Z}_n\right) \otimes C \xrightarrow{\cong} \operatorname{Tot}(\mathbb{Z}(X) \otimes C).$$

We can assume that C is cofibrant, even free in each degree.

Form cofibrant chain complexes $F_k C$ and maps $F_k C \rightarrow C$ such that the maps $H_k F_k C \rightarrow H_k C$ are isomorphisms, and such that $H_p(F_k C) = 0$ for $p \neq k$.

It follows that there is a chain homotopy equivalence

$$\bigoplus_{k \geq 0} F_k C \xrightarrow{\cong} C.$$

There are isomorphisms

$$H_p(\tilde{Z}_n \otimes F_k C) \cong \begin{cases} H_n(X) \otimes H_k(C) & \text{if } p = n + k, \\ \text{Tor}(H_n(X), H_k(C)) & \text{if } p = n + k + 1, \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Do you need a spectral sequence?

Hint: Filter $F_k C$.

Adding up these isomorphisms gives split short exact sequences

$$\begin{aligned} 0 \rightarrow H_n(X) \otimes H_k(C) \rightarrow H_{n+k} \text{Tot}(\tilde{Z}_n \otimes C) \\ \rightarrow \text{Tor}(H_n(X), H_{k-1}(C)) \rightarrow 0 \end{aligned} \quad (10)$$

for $k \geq 0$, and $H_k \text{Tot}(\tilde{Z}_n \otimes C) = 0$ for $k < 0$.

Taking a direct sum of the sequences (10) (and

reindexing) gives short exact sequences

$$\begin{aligned}
0 \rightarrow \bigoplus_{0 \leq p \leq n} H_{n-p}(X) \otimes H_p(C) &\rightarrow H_n \text{Tot}(\mathbb{Z}(X) \otimes C) \\
&\rightarrow \bigoplus_{0 \leq q \leq n-1} \text{Tor}(H_{n-1-q}(X), H_q(C)) \rightarrow 0
\end{aligned} \tag{11}$$

The sequence (11) and the Eilenberg-Zilber Theorem (Theorem 26.1) together imply the following result:

Theorem 27.2 (Künneth Theorem). *Suppose X and Y are simplicial sets. Then there is a natural short exact sequence*

$$\begin{aligned}
0 \rightarrow \bigoplus_{0 \leq p \leq n} H_{n-p}(X) \otimes H_p(Y) &\rightarrow H_n(X \times Y) \\
&\rightarrow \bigoplus_{0 \leq q \leq n-1} \text{Tor}(H_{n-1-q}(X), H_q(Y)) \rightarrow 0.
\end{aligned}$$

This sequence splits, but not naturally.

The coefficient ring \mathbb{Z} in the statement of Theorem 27.2 can be replaced by a principal ideal domain R . The same theorem holds for $H_*(X \times Y, R)$, with the same proof.

If $R = F$ is a field, all F -modules are free and the Tor terms in the Theorem vanish, so

$$H_n(X \times Y, F) \cong \bigoplus_{0 \leq p \leq n} H_{n-p}(X, F) \otimes_F H_p(Y, F). \quad (12)$$

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