Lecture 007 (April 13, 2011)

16 Abelian category localization

Suppose that $\mathbf{A}$ is a small abelian category, and let $\mathbf{B}$ be a full subcategory such that in every exact sequence

$$0 \to a' \to a \to a'' \to 0$$

in $\mathbf{A}$, a is an object of $\mathbf{C}$ if and only if b' and a" are objects of $\mathbf{B}$. Such a subcategory $\mathbf{B}$ is said to be thick (or dense, or a Serre subcategory). Observe that $\mathbf{B}$ is closed under taking subobjects, quotients and finite direct sums.

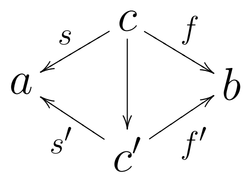
It is common to write Σ for the set of morphisms of $\mathbf{A}$ with kernel and cokernel in $\mathbf{B}$. Then the quotient category

$$\mathbf{A}/\mathbf{B} = \mathbf{A}(\Sigma^{-1})$$

is constructed by formally inverting the morphisms of Σ. The category $\mathbf{A}/\mathbf{B}$ is constructed from a calculus of fractions: it has the same objects as $\mathbf{A}$, and a morphism $[s, f] : a \to b$ of $\mathbf{A}/\mathbf{B}$ is an equivalence class of maps

$$a \xleftarrow{s} c \xrightarrow{f} b,$$

where $s \in \Sigma$. The equivalence relation is generated by commutative diagrams



The category Σ/b of morphisms $c \to b$ in Σ is filtered (this is the calculus of fractions part), and there is an identification

$$\hom_{\mathbf{A}/\mathbf{B}}(a, b) = \varinjlim_{c \xrightarrow{s} a \in \Sigma} \hom_{\mathbf{A}}(c, b).$$

Composition of morphisms in $\mathbf{A}/\mathbf{B}$ is defined by pullback. This works, because the set Σ is preserved by pullback. Note that

$$[f, s] = [1, f] \cdot [s, 1],$$

and that $[1, s]$ is the inverse of $[s, 1]$ in $\mathbf{A}/\mathbf{B}$. It follows that

$$[f, s] \cdot [1, s] = [1, f].$$

There is a canonical functor

$$\pi : \mathbf{A} \to \mathbf{A}/\mathbf{B}$$

which is the identity on objects, and sends a morphism $f : a \to b$ to the morphism $[1, f]$ of $\mathbf{A}/\mathbf{B}$.

This functor π satisfies a universal property: every functor $g : \mathbf{A} \to \mathbf{C}$ which inverts the morphisms of Σ has a unique factorization $g_* : \mathbf{A}/\mathbf{B} \to \mathbf{C}$ through π.

The category $\mathbf{A}/\mathbf{B}$ is abelian, and the functor π is exact.

The following (Theorem 16.1) is Quillen's Localization Theorem for abelian categories. It first appeared in [2], and that is still one of the better writeups of the result. It implies a localization theorem for the K-theory of coherent sheaves, which will be discussed below. The K-theory of coherent sheaves coincides with ordinary vector bundle K-theory for regular schemes, on account of the Resolution Theorem (Theorem 14.3, Corollary 14.4), so that Theorem 16.1 implies a localization result for the K-theory of regular schemes. There is a more recent result for the K-theory of perfect complexes which has far reaching consequences for the K-theory of singular schemes, which is due to Thomason and Trobaugh [3]. The Thomason-Trobaugh result is delicate, and will not be discussed in this course — it is nevertheless that last available word on this subject.

Theorem 16.1. *Suppose that* $\mathbf{B}$ *is a Serre subcategory of a small abelian category* $\mathbf{A}$*, let* $i : \mathbf{B} \to \mathbf{A}$ *be the corresponding inclusion functor, and let* $\pi : \mathbf{A} \to \mathbf{A}/\mathbf{B}$ *be the quotient functor. Then the functors* i *and* π *induce a fibre homotopy sequence*

$$BQ(\mathbf{B}) \xrightarrow{i_*} BQ(\mathbf{A}) \xrightarrow{\pi_*} BQ(\mathbf{A}/\mathbf{B}).$$

Write $i : Q(\mathbf{B}) \to Q(\mathbf{A})$ and $\pi : Q(\mathbf{A}) \to Q(\mathbf{A}/\mathbf{B})$ for the induced functors on Q-constructions. The idea of the proof of the Theorem is to show that

1) the canonical functor $Q(\mathbf{B}) \to 0/\pi$ is a weak equivalence, and
2) every morphism $u : a \to b$ in $Q(\mathbf{A}/\mathbf{B})$ induces a weak equivalence

$$b/\pi \xrightarrow{\simeq} a/\pi.$$

By a duality argument, it suffices to prove statement 2) for morphisms $u = i_!$ arising from monomorphisms i. In particular, it suffices to prove 2) for the morphisms $i_{b!}$ associated to maps $0 \to b$.

Suppose that $b \in \mathbf{A}$. The category E_b has as objects all maps $h : a \to b$ of $\mathbf{A}$ which are in the set Σ of morphisms which induce isomorphisms in $\mathbf{A}/\mathbf{B}$. The morphisms $h \to h'$ of E_b are equivalence classes of pictures

$$\begin{array}{ccc} c & \overset{i}{\rightarrowtail} & d \\ {\scriptstyle p}\downarrow & \overset{h''}{\searrow} & \downarrow{\scriptstyle h'} \\ a & \underset{h}{\rightarrow} & b \end{array}$$

where the equivalence relation is defined as in the formation of the category $Q\mathbf{A}$, and composition is defined by pullback. Observe that the morphisms p and i must be in Σ, since $\ker(p) \subset \ker(h')$, and so $\pi(i)$ is an isomorphism of $\mathbf{A}/\mathbf{B}$.

There is a functor $k_b : E_b \to Q(\mathbf{B})$ which is defined by taking the map defined by the picture above to the map defined by the picture

$$\ker(h) \twoheadleftarrow \ker(h'') \rightarrowtail \ker(h').$$

Let F_b be the full subcategory of b/π whose objects are isomorphisms $\theta : b \to \pi(a)$ (recall that isomorphisms of $Q(\mathbf{M})$ are isomorphisms of $\mathbf{M}$, for any exact category $\mathbf{M}$).

The subcategory b/Σ of morphisms $b \to a$ in Σ is filtering.

There is a functor

$$E : b/\Sigma \to \mathbf{cat}$$

which takes the morphism

$$\begin{array}{ccc} & & a \\ & \overset{u}{\nearrow} & \\ b & & \downarrow f \\ & \underset{v}{\searrow} & \\ & & a' \end{array}$$

to the functor $f_* : E_a \to E_{a'}$. There is also, for each $s : b \to a$ in Σ, a functor

$$s^* : E_a \to F_b$$

which takes the object $h : d \to a$ to the object $\pi(s)^{-1}\pi(h)$ of F_b. Then it's relatively easy to show that the functors s^*, $s : b \to a$ in Σ together induce a functor

$$\phi : \varinjlim_{b \to a \in \Sigma} E_a \to F_b,$$

and that the functor ϕ is an isomorphism.

The following diagram commutes up to homotopy for any morphism $s : b \to a$ in Σ:

$$\begin{array}{ccccc} E_a & \xrightarrow{s^*} & F_b & \xrightarrow{i} & b/\pi \\ {\scriptstyle k_a}\downarrow & & & & \downarrow{\scriptstyle i^*_{b!}} \\ Q(\mathbf{B}) & \xrightarrow[\cong]{} & F_0 & \xrightarrow[i]{} & 0/\pi \end{array} \tag{1}$$

where the morphisms labelled by i are canonical inclusions, and the indicated isomorphism of categories is defined by sending an object c to the isomorphism $i_{\pi(c)!} : 0 \xrightarrow{\cong} \pi(c)$. The functor $E_a \to 0/\pi$ across the top sends $h : c \to a$ to the composite

$$0 \xrightarrow{i_{b!}} b \xrightarrow{\pi(h)^{-1}\pi(s)} c$$

or $i_{c!} : 0 \to c$, while the functor across the bottom sends h to the map $i_{\ker(h)!} : 0 \to \ker(h)$, and the map $\ker(h) \to c$ defines the homotopy.

To prove Theorem 16.1, it suffices to prove the following:

Lemma 16.2. *The map $i : F_b \to b/\pi$ is a weak equivalence for all $b \in \mathbf{A}/\mathbf{B}$.*

Lemma 16.3. *The map $k_a : E_a \to Q(\mathbf{B})$ is a weak equivalence for all $a \in \mathbf{A}$.*

Lemma 16.4. *Suppose that $g : b \to b'$ is a morphism of Σ. Then the induced functor $E_b \to E_{b'}$ is a weak equivalence.*

It follows from Lemma 16.4 that F_b is a filtered colimit of categories, each of which is canonically equivalent to E_b, and so all maps $s^* : E_a \to F_b$ are weak equivalences. We therefore know that all morphisms other than $i_{b!}^*$ in the diagram (1) are weak equivalences, so $i_{b!}^*$ is a weak equivalence too. This would complete the proof of Theorem 16.1.

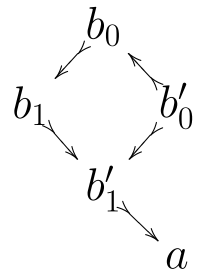
Proof of Lemma 16.2. This is somehow the key point.

Suppose that $u : b \to \pi(a)$ is an object of b/π. We show that the category i/u is contractible.

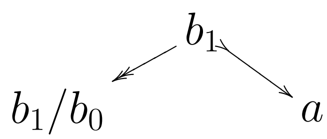
We need a concept: an admissible layer in a for an object a of an exact category $\mathbf{M}$ is a sequence (b_0, b_1) of subobjects

$$b_0 \rightarrowtail b_1 \rightarrowtail a$$

Say that $(b_0, b_1) \leq (b_0', b_1')$ if there is a relation



Then the assignment which takes a layer (b_0, b_1) to the morphism of $Q(\mathbf{M})$ which is defined by the picture



(subject to making choices) defines a functor $L(a) \to Q(\mathbf{M})/a$, and one can show that this functor is an equivalence of categories. Observe that any two layers (b_0, b_1) and (b_0', b_1') of a have a least upper bound

$$(b_0 \cap b_0, b_1 + b_1')$$

so that the category $L(a)$ is filtered.

Suppose that $u : b \to \pi(a)$ is defined by the admissible layer

$$b : \ b_0 \rightarrowtail b_1 \rightarrowtail a$$

of subobjects in $\mathbf{A}/\mathbf{B}$ (with $b = b_1/b_0$). The category i/u is equivalent to the category $L_v(a)$ of layers

$$c : \ c_0 \rightarrowtail c_1 \rightarrowtail a$$

in a in the category $\mathbf{A}$ such that $\pi(c) = b$. The functor π is exact, so preserves least upper bounds in layers. It follows that $L_v(a)$ is filtered, and is therefore contractible. □

Proof of Lemma 16.3. Let E_a' be the full subcategory of E_a whose objects are the epimorphisms $c \twoheadrightarrow a$ of Σ. The objects of E_a' may therefore be identified with exact sequences

$$E : \ 0 \to d \rightarrowtail c \overset{p}{\twoheadrightarrow} a \to 0$$

of $\mathbf{A}$ with $d \in \mathbf{B}$. Observe that the functor f_a defined by the composite

$$E_a' \subset E_a \xrightarrow{k_a} Q(\mathbf{B})$$

sends the exact sequence above to the kernel object d.

The proof now comes in two parts:

a) show that the functor f_a is a weak equivalence, and

b) show that the inclusion $E'_a \subset E_a$ is a weak equivalence.

Statement a) is proved by showing that all categories f_a/x are contractible. In effect, every morphism

$$\theta : d = f_a(E) \to x$$

in $Q(\mathbf{B})$ has a factorization $\theta = p^! j_!$ where j is a monomorphism and p is an epimorphism. Write C' for the subcategory of f_a/x whose objects are morphisms $q^! : f_a(E') \to x$ which are induced by epimorphisms $q : x \twoheadrightarrow f_a(E')$ of $\mathbf{B}$. Then the pushout diagrams

$$\begin{array}{ccccc} d & \rightarrowtail & c & \overset{p}{\twoheadrightarrow} & a \\ {\scriptstyle j}\downarrow & & \downarrow & & \\ d' & \rightarrowtail & \overline{c} & & \end{array}$$

define a functor $f_a/x \to C'$ which is left adjoint to the inclusion $C' \subset f_a/x$, and so the inclusion $C' \subset f_a/x$ is a weak equivalence. The category C'

has an initial object, defined by the picture

$$
\begin{array}{ccccc}
0 & \rightarrowtail & a & \xrightarrow{1} \!\!\!\!\rightarrow & a \\
{\scriptstyle p_x^!}\downarrow & & & & \\
x & & & &
\end{array}
$$

so that C' is contractible.

For statement b), let $\text{Mon}_\Sigma(a)$ be the category whose objects are monomorphisms $e \rightarrowtail a$ in Σ, and whose morphisms are commutative triangles of admissible monomorphisms. Then it's easy to see that $\text{Mon}_\Sigma(a)$ is filtered: any two objects $e \rightarrowtail a$ and $e' \rightarrowtail a$ of $\text{Mon}_\Sigma(a)$ have an upper bound $e + e' \rightarrowtail a$.

There is a functor $\text{im} : E_a \to \text{Mon}_\Sigma(a)$ which is defined by taking a morphism $h : b \to a$ of Σ to its image $\text{im}(h) \rightarrowtail a$. For each $i : k \rightarrowtail a$ in Σ there is a functor

$$F_k : E'_k \to k/\,\text{im}$$

which is defined by taking the epi $p : d \twoheadrightarrow k$ in Σ to the diagram

$$
\begin{array}{ccccc}
 & & & & d \\
 & & & & \downarrow{\scriptstyle p} \\
k & & \xrightarrow{\;\;1\;\;} & & k \\
 & {\scriptstyle i}\searrow & & \swarrow{\scriptstyle i} & \\
 & & a & &
\end{array}
$$

This functor F_k has a right adjoint, which is essentially defined by pullback, and is therefore a weak equivalence.

Pulling back along a monomorphism

$$k' \stackrel{i}{\rightarrowtail} k \rightarrowtail a$$

in $\mathrm{Mon}_\Sigma(a)$ defines a functor

$$i^* : E'_k \to E'_{k'}.$$

This functor i^* commutes with taking kernels (namely with the functors f_k and $f_{k'}$) up to natural isomorphism, and is therefore a weak equivalence. There is also a homotopy commutative diagram

$$\begin{array}{ccc} E'_k & \xrightarrow{i^*} & E'_{k'} \\ {\scriptstyle F_k}\downarrow & & \downarrow{\scriptstyle F_{k'}} \\ k/\operatorname{im} & \xrightarrow[i^*]{} & k/\operatorname{im} \end{array}$$

so that the maps $i^* : k/\operatorname{im} \to k'/\operatorname{im}$ are weak equivalences. But this means that the sequence

$$E'_a \to E_a \to \mathrm{Mon}_\Sigma(a)$$

is a homotopy fibre sequence. The category $\mathrm{Mon}_\Sigma(a)$ is contractible, so that the inclusion functor $E'_a \subset E_a$ is a weak equivalence. $\square$

Proof of Lemma 16.4. Suppose that $g : b \to b'$ is a morphism of Σ. The diagram

$$\begin{array}{ccc} E_b & \xrightarrow{g_*} & E_{b'} \\ & {\scriptstyle k_b}\searrow \quad \swarrow{\scriptstyle k_{b'}} & \\ & Q(\mathbf{B}) & \end{array}$$

is homotopy commutative, with homotopy determined by the monomorphisms $\ker(h) \rightarrowtail \ker(gh)$. The maps k_b and $k_{b'}$ are weak equivalences by Lemma 16.3, so that g_* is a weak equivalence as well. $\square$

17 Coherent sheaves and open subschemes

1) Suppose that X is a Noetherian scheme and that U is an open subscheme of X. Write $j : U \hookrightarrow X$ for the inclusion of U in X. Write Z for the complement $Z = X - U$ with the reduced subscheme structure, and let $i : Z \hookrightarrow X$ denote the corresponding closed immersion. The "kernel" $\mathbf{M}_{X-U}$ of the restriction map

$$j^* : \mathbf{M}(X) \to \mathbf{M}(U)$$

in coherent sheaves consists of all those modules M such that $j^*(M) = M|_U$ are zero objects, and as such consists of those modules which are supported on $X-U$ in the sense that, in stalks, $M_x \cong 0$ for $x \in U$. Alternatively, $M \in \mathbf{M}_{X-U}$ if and only if there is some power I^n of the defining sheaf of ideals for Z such that $I^n M = 0$. The category $\mathbf{M}_{X-U}$ is a Serre subcategory of $\mathbf{M}(X)$, and it's an exercise to show that the induced functor

$$\mathbf{M}(X)/\mathbf{M}_{X-U} \to \mathbf{M}(U)$$

is an equivalence of categories.

The category $\mathbf{M}(Z)$ of coherent sheaves on Z can be identified up to equivalence with those modules on X which are annihilated by I, via the transfer map

$$\mathbf{M}(Z) \xrightarrow{i_*} \mathbf{M}(X).$$

The resulting functor

$$\mathbf{M}(Z) \to \mathbf{M}_{X-U}$$

induces a stable equivalence

$$K(\mathbf{M}(Z)) \simeq K(\mathbf{M}_{X-U})$$

by dévissage (Theorem 15.1), and it follows from Theorem 16.1 that there is a homotopy fibre sequence of (symmetric) spectra

$$K(\mathbf{M}(Z)) \xrightarrow{i_*} K(\mathbf{M}(X)) \xrightarrow{j^*} K(\mathbf{M}(U)),$$

and a corresponding fibre sequence

$$K'(Z) \xrightarrow{i_*} K'(X) \xrightarrow{j^*} K'(U)$$

of stably fibrant models. It follows that there is a long exact sequence

$$\begin{gathered} \dots \xrightarrow{\partial} K'_q(Z) \xrightarrow{i_*} K'_q(X) \xrightarrow{j^*} K'_q(U) \to \\ \dots \to K'_1(U) \xrightarrow{\partial} K'_0(Z) \xrightarrow{i_*} K'_0(X) \xrightarrow{j^*} K'_0(U) \to 0 \end{gathered}$$

Note the surjectivity of the map $j^* : K'_0(X) \to K'_0(U)$.

2) Suppose that $U \subset \mathrm{Sp}(\mathbb{Z})$ is an open subset. Then the reduced closed complement

$$Z = \mathrm{Sp}(\mathbb{Z}) - U$$

can be identified with the scheme

$$\mathrm{Sp}(\mathbb{F}_{p_1}) \sqcup \dots \sqcup \mathrm{Sp}(\mathbb{F}_{p_n})$$

for some finite collection of primes $\{p_1, \dots, p_n\}$, and so there is an equivalence

$$\mathbf{M}(Z) \simeq \mathbf{M}(\mathbb{F}_{p_1}) \times \dots \times \mathbf{M}(\mathbb{F}_{p_n}).$$

Then there is a long exact sequence

$$\dots \to K'_1(U) \xrightarrow{\partial} \bigoplus_{i=1}^{n} K'_0(\mathbb{F}_{p_i}) \xrightarrow{i_*} K'_0(\mathbb{Z}) \xrightarrow{j^*} K'_0(U) \to 0.$$

Taking a filtered colimit of these fibre sequences over all open subsets $U \subset \mathrm{Sp}(\mathbb{Z})$ gives a long exact sequence

$$\cdots \to K_1'(\mathbb{Q}) \xrightarrow{\partial} \bigoplus_p K_0'(\mathbb{F}_p) \xrightarrow{i_*} K_0'(\mathbb{Z}) \xrightarrow{j^*} K_0'(\mathbb{Q}) \to 0$$

where the direct sum is indexed over all prime numbers p. All of the rings appearing in this exact sequence are regular, so that the sequence can be rewritten as a K-theory exact sequence

$$\cdots \to K_1(\mathbb{Q}) \xrightarrow{\partial} \bigoplus_p K_0(\mathbb{F}_p) \xrightarrow{i_*} K_0(\mathbb{Z}) \xrightarrow{j^*} K_0(\mathbb{Q}) \to 0$$

Similarly, if A is any Dedekind domain (such as a ring of integers in a number field, or the ring of functions of any smooth affine curve over a field), there is a long exact sequence

$$\cdots \to K_1(k(A)) \xrightarrow{\partial} \bigoplus_{\mathcal{P} \in \mathrm{Sp}(A)} K_0(A/\mathcal{P}) \\ \xrightarrow{i_*} K_0(A) \xrightarrow{j^*} K_0(k(A)) \to 0$$

where $k(A)$ is the quotient field of A.

NB: This long exact sequence is also the localization sequence associated to the functor

$$j^* : \mathbf{M}(A) \to \mathbf{M}(k(A))$$

which is defined by localization at the generic point; this observations specializes to $A = \mathbb{Z}$.

3) Suppose that R is a discrete valuation ring with quotient field $k(R)$ and residue field k (the examples include Witt rings, so the characteristics could be mixed). The kernel of the localization map

$$\mathbf{M}(R) \to \mathbf{M}(k(R))$$

is the collection of finitely generated R modules which are annihilated by some power π^n of the uniformizing parameter π (aka. generator of the maximal ideal of R). It follows that there is a fibre sequence

$$K(k) \xrightarrow{p_*} K(R) \xrightarrow{j^*} K(k(R))$$

and hence a long exact sequence

$$\cdots \to K_1(R) \to K_1(k(R)) \xrightarrow{\partial} K_0(k)$$
$$\xrightarrow{p_*} K_0(R) \xrightarrow{j^*} K_0(k(R)) \to 0.$$

The ring R is local, so that all finitely generated projective R-modules are free, and so $K_0(R) \cong \mathbb{Z}$. The map $K_0(R) \to K_0(k(R))$ is isomorphic to the identity map $\mathbb{Z} \to \mathbb{Z}$, and so we have an exact sequence

$$K_1(R) \to K_1(k(R)) \xrightarrow{\partial} \mathbb{Z} \to 0$$

Since R is local, the group $Sl(R)$ is generated by elementary transformation matrices, so that this sequence can be identified up to isomorphism with the sequence

$$R^* \to k(R)^* \xrightarrow{v} \mathbb{Z} \to 0$$

which is defines the valuation v.

4) Suppose that the Noetherian scheme X has Krull dimension 1 over an algebraically closed field k, and let $j : U \subset X$ be an open subscheme. The reduced complement Z is finite over k and there is an isomorphism

$$Z \cong \mathrm{Sp}(k) \sqcup \cdots \sqcup \mathrm{Sp}(k).$$

Then there is a long exact sequence

$$\ldots K_1'(U) \xrightarrow{\partial} \bigoplus_{i=1}^{n} K_0'(k) \xrightarrow{i_*} K_0'(X) \to K_0'(U) \to 0.$$

The transfer map i_* is a sum $\sum i_{x*}$ of the transfer maps corresponding to the points $x \in Z$.

If X is irreducible, then taking a filtered colimit of these sequences over all U open in X gives a long exact sequence

$$\ldots K_1'(k(X)) \xrightarrow{\partial} \bigoplus_{x \in X} K_0'(k) \xrightarrow{\sum i_{x*}} K_0'(X) \to K_0'(k(X)) \to 0$$

where $k(X)$ is the function field of X.

Finally, if X is a smooth curve over k then all of the local rings $\mathcal{O}_{x,X}$ are discrete valuation rings, and there is a comparison of localization sequences

$$\begin{array}{ccccc} K_1(X) & \longrightarrow & K_1(k(X)) & \xrightarrow{\partial} & \bigoplus_{x\in X} K_0(k) \\ \downarrow & & \downarrow = & & \downarrow pr_x \\ K_1(\mathcal{O}_{x,X}) & \longrightarrow & K_1(k(X)) & \xrightarrow{\partial} & K_0(k) \\ \cong\downarrow & & \downarrow\cong & & \downarrow\cong \\ \mathcal{O}^*_{x,X} & \longrightarrow & k(X)^* & \xrightarrow[v_x]{} & \mathbb{Z} \end{array}$$

It follows that the boundary map

$$\partial : K_1(k(X)) \to \bigoplus_{x\in X} K_0(k) \cong \bigoplus_{x\in X} \mathbb{Z}$$

can be identified with sum $\sum_{x\in X} v_x$ of the valuation maps v_x. It follows as well that there is an exact sequence

$$K_1(k(X)) \xrightarrow{\partial} \bigoplus_{x\in X} K_0(k) \to \mathrm{Cl}(X) \to 0$$

where $\mathrm{Cl}(X)$ is the divisor class group of X. If X is also separated then there is an exact sequence

$$K_1(k(X)) \xrightarrow{\partial} \bigoplus_{x\in X} K_0(k) \to \mathrm{Pic}(X) \to 0$$

where

$$\mathrm{Pic}(X) = H^1_{et}(X, \mathbb{G}_m)$$

is the Picard group of X.

18 Product formulas

As usual, we begin with a little homotopy theory. Suppose given a diagram of pointed maps

$$\begin{array}{ccc} Z \wedge X_1 & \longrightarrow & X_2 \\ {\scriptstyle 1\wedge f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Z \wedge Y_1 & \longrightarrow & Y_2 \end{array} \tag{2}$$

There is a model structure on the category of arrows of pointed maps for which a map

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow[\alpha_2]{} & B_2 \end{array}$$

from f_1 to f_2 is a weak equivalence (respectively cofibration) if and only if the maps α_1 and α_2 are weak equivalences (respectively cofibrations). It's not hard to see that an object $f : X \to Y$ in this category is fibrant if and only if Y is fibrant and f is a fibration. It follows that by taking fibrant models for both f_1 and f_2, the square (2) can be replaced up to weak equivalence by a square for which the maps f_1 and f_2 are fibrations and the spaces X_1, X_2, Y_1 and Y_2 are fibrant. Suppose henceforth that the maps and spaces in the diagram (2) satisfy these criteria.

Let F_1 and F_2 be the fibres of f_1 and f_2, respectively, over the respective base points of Y_1 and Y_2. Then there is a unique induced map $m : Z \wedge F_1 \to F_2$ such that the diagram

$$\begin{array}{ccc} Z \wedge F_1 & \xrightarrow{m} & F_2 \\ {\scriptstyle 1\wedge i}\downarrow & & \downarrow{\scriptstyle i} \\ Z \wedge X_1 & \xrightarrow[m]{} & X_2 \end{array}$$

commutes.

The map $m : Z \wedge Y_1 \to Y_2$ induces an action

$$m : Z \wedge \mathbf{hom}_*(K, Y_1) \to \mathbf{hom}_*(K, Y_2)$$

for any pointed space K: this map m is adjoint to the composite

$$Z \wedge \mathbf{hom}_*(K, Y_1) \wedge K \xrightarrow{1\wedge ev} Z \wedge Y_1 \xrightarrow{m} Y_2,$$

where $ev : \mathbf{hom}_*(K, Y_1) \wedge K \to Y_1$ is the evaluation map. This induced pairing is natural in K.

It follows that there is an induced pairing

$$\begin{array}{ccc} Z \wedge PY_1 & \xrightarrow{m} & PY_2 \\ {\scriptstyle 1\wedge \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z \wedge Y_1 & \xrightarrow[m]{} & Y_2 \end{array}$$

for the path-loop fibration π, and there is a commutative diagram

$$\begin{array}{ccc}
Z \wedge F_1 & \xrightarrow{m} & F_2 \\
{\scriptstyle 1\wedge v_1}\downarrow \simeq & & \simeq \downarrow {\scriptstyle v_2} \\
Z \wedge (PY_1 \times_{Y_1} X_1) & \longrightarrow & PY_2 \times_{Y_2} X_2 \\
{\scriptstyle 1\wedge u_1}\uparrow & & \uparrow {\scriptstyle u_2} \\
Z \wedge \Omega Y_1 & \xrightarrow[m]{} & \Omega Y_2
\end{array}$$

The map $v_i^{-1} u_i$ is the boundary homomorphism $\partial : \Omega Y_i \to F_i$ in the pointed homotopy category, and it follows that there is a commutative diagram

$$\begin{array}{ccc}
Z \wedge \Omega Y_1 & \xrightarrow{m} & \Omega Y_2 \\
{\scriptstyle 1\wedge\partial}\downarrow & & \downarrow {\scriptstyle \partial} \\
Z \wedge F_1 & \xrightarrow[m]{} & F_2
\end{array}$$

in the homotopy category.

Generally, a pairing $m : Z \wedge X \to X$ induces a map

$$\cup : \pi_p(Z) \otimes \pi_q(X) \to \pi_{p+q}(X).$$

In effect, if $\alpha : S^p \to Z$ and $\beta : S^q \to X$ represent elements $[\alpha] \in \pi_p(Z)$ and $[\beta] \in \pi_q(X)$ respectively, then $[\alpha] \cup [\beta]$ is represented by the composite

$$S^{p+q} \cong S^p \wedge S^q \xrightarrow{\alpha\wedge\beta} Z \wedge X \xrightarrow{m} X.$$

Here, S^r is the r-fold smash power $S^1 \wedge \cdots \wedge S^1$ of copies of S^1, and it's an exercise to show that the homotopy group $\pi_r(X)$ for a pointed Kan complex X is isomorphic to the set $\pi_*(S^r, X)$ of pointed homotopy classes of maps of pointed simplicial sets from S^r to X.

Suppose that $\alpha : S^p \to Z$ and $\beta : S^{q+1} \to Y_1$ represent elements of the groups $\pi_p(Z)$ and $\pi_{q+1}(Y_1)$ respectively. The boundary map

$$\partial : \pi_{q+1}(Y_1) \to \pi_q(F_1)$$

is defined for $[\beta]$ by taking the adjoint $\beta_* : S^q \to \Omega Y_1$ and forming the composite

$$S^q \xrightarrow{\beta_*} \Omega Y_1 \xrightarrow{\partial} F_1$$

in the pointed homotopy category:

$$\partial([\beta]) = \partial \cdot [\beta_*].$$

The composite

$$S^{p+q} \cong S^p \wedge S^q \xrightarrow{\alpha \wedge \beta_*} Z \wedge \Omega Y_1 \xrightarrow{m} \Omega Y_2$$

is adjoint to the composite

$$S^{p+q+1} \cong S^p \wedge S^{q+1} \xrightarrow{\alpha \wedge \beta} Z \wedge Y_1 \xrightarrow{m} Y_2.$$

It follows that

$$\partial([\alpha] \cup [\beta]) = [\alpha] \cup \partial([\beta]). \tag{3}$$

Now suppose that X is a Noetherian scheme. Then tensor product defines a biexact pairing

$$\otimes : \mathcal{P}(X) \times \mathbf{M}(X) \to \mathbf{M}(X). \tag{4}$$

By Proposition 9.3 (Lecture 004), the biexact pairing (4) induces a smash product pairing

$$K(\mathcal{P}(X)) \wedge_\Sigma K(\mathbf{M}(X)) \xrightarrow{\cup} K(\mathbf{M}(X))$$

of symmetric spectra, but we have to be a little careful to interpret it properly. The pairing must be derived in the stable category, because the smash product doesn't quite preserve stable equivalences.

Generally, if $m : Y_1 \wedge_\Sigma Y_2 \to Y_3$ is a morphism of symmetric spectra, then by taking stably fibrant models $j_V : V \to FV$ and stably cofibrant models $\pi_W : CW \to W$ one constructs a diagram

$$\begin{array}{ccc} Y_1 \wedge_\Sigma Y_2 & \xrightarrow{m} & Y_3 \\ {\scriptstyle \pi_{Y_1} \wedge \pi_{Y_2}} \uparrow & & \uparrow {\scriptstyle \pi_{Y_3}} \\ CY_1 \wedge_\Sigma CY_2 & \xrightarrow{m'} & CY_3 \\ {\scriptstyle j_{CY_1} \wedge j_{CY_2}} \downarrow & & \downarrow {\scriptstyle j_{CY_3}} \\ FCY_1 \wedge_\Sigma FCY_2 & \xrightarrow[m'']{} & FCY_3 \end{array} \tag{5}$$

where the map m' exists because $CY_1 \wedge CY_2$ is stably cofibrant, and m'' exists because $j_{CY_1} \wedge j_{cY_2}$

is a stably trivial cofibration (see [1, Prop. 4.19], for example). The maps π_{Y_i} and j_{CY_i} can be chosen functorially because the stable model structure on symmetric spectra is cofibrantly generated, and the maps m' and m'' are uniquely determined up to simplicial homotopy. The induced maps $\pi_* : FCY_i \to FY_i$ are stable hence levelwise weak equivalences of stably fibrant symmetric spectra, and so the objects FCY_i are stably fibrant models for the objects Y_i, respectively.

Recall that the functor $\mathrm{Spt}_\Sigma \to s\mathbf{Set}_*$ which takes a symmetric spectrum X to the pointed space X^n at level n has a left adjoint

$$F_n : s\mathbf{Set}_* \to \mathrm{Spt}_\Sigma .$$

One way to define this functor is to set

$$F_n(K) = V(\Sigma^\infty K[n]),$$

where $V : \mathrm{Spt} \to \mathrm{Spt}_\Sigma$ is the left adjoint to the functor $U : \mathrm{Spt}_\Sigma \to \mathrm{Spt}$ which forgets the symmetric group actions. The functor V preserves cofibrations, so that all symmetric spectra $F_n(K)$ are cofibrant. It follows that if $j_X : X \to FX$ is a stably fibrant model for a symmetric spectrum X, then there are isomorphisms

$$[F_n(K), X] \cong [F_n(K), FX] \cong \pi(F_n(K), FX)$$
$$\cong \pi(K, FX^n) \cong [\Sigma^\infty K[n], UFX]$$

In particular, a map $f : X \to Y$ of symmetric spectra is a stable equivalence if and only if the induced maps

$$[F_n(S^r), X] \xrightarrow{f_*} [F_n(S^r), Y]$$

are group isomorphisms for all n and r. This requirement is over determined: it suffices that f_* be an isomorphism in the cases where $r = 0$ if $n > 0$ and for all r if $n = 0$, since there are stable equivalences of spectra

$$\Sigma^\infty(S^r)[n] \to \Sigma^\infty(S^{r-1})[n-1].$$

Write

$$\pi_n^s(X) = \begin{cases} [F_0(S^n), X] & \text{if } n \geq 0, \text{ and} \\ [F_{-n}(S^0, X] & \text{if } n < 0. \end{cases}$$

Then a map $f : X \to Y$ of symmetric spectra is a stable equivalence if and only if the induced maps

$$f_* : \pi_n^s(X) \to \pi_n^s(Y)$$

are isomorphisms for all $n \in \mathbb{Z}$.

Note that the stable homotopy groups $\pi_n^s(X)$ coincide up to natural isomorphism with the traditional stable homotopy groups $\pi_n^s(UF(X))$ of the spectrum $UF(X)$ underlying a stably fibrant model $F(X)$ of X.

There are natural isomorphisms

$$F_n(K) \wedge_\Sigma F_m(L) \cong F_{n+m}(K \wedge L)$$

(see [1, Cor. 4.18]). From the diagram (5) above, we see that any smash product pairing

$$m : Y_1 \wedge_\Sigma Y_2 \to Y_3$$

induces pairings

$$\begin{array}{ccc}
[F_n(K), Y_1] \otimes [F_m(L), Y_2] & \overset{\cup}{\dashrightarrow} & [F_{n+m}(K \wedge L), Y_3] \\
\cong \downarrow & & \downarrow \cong \\
[F_n(K), FCY_1] \otimes [F_m(L), FCY_2] & & [F_{n+m}(K \wedge L), FCY_3] \\
\cong \downarrow & & \downarrow \cong \\
\pi(F_n(K), FCY_2) \otimes \pi(F_m(L), FCY_2) & \xrightarrow[m''_*]{} & \pi(F_{n+m}(K \wedge L), FCY_3)
\end{array}$$

where the pairing m''_* takes the pair $([\alpha], [\beta])$ to map represented by the composite

$$F_{n+m}(K \wedge L) \cong F_n(K) \wedge_\Sigma F_m(L) \xrightarrow{\alpha \wedge \beta} FCY_1 \wedge_\Sigma FCY_2 \xrightarrow{m''} FCY_3.$$

In this way, we see that the smash product pairing m induces a cup product pairing

$$\pi_n^s(Y_1) \otimes \pi_m^s(Y_2) \xrightarrow{\cup} \pi_{n+m}^s(Y_3). \qquad (6)$$

If all symmetric spectra Y_i are connective, then there are isomorphisms

$$\pi_n^s(Y_i) \cong \pi_n(FCY_i^0),$$

and the pairing (6) is isomorphic to the pairing

$$\pi_n(FCY_1^0) \otimes \pi_m(FCY_2^0) \xrightarrow{\cup} \pi_{n+m}(FCY_3^0)$$

which is induced by the space-level smash product pairing

$$FCY_1^0 \wedge FCY_2^0 \to FCY_3^0$$

which is a component of the map of symmetric spectra m''. Observe also that the component

$$FCY_1^1 \wedge FCY_2^1 \to FCY_3^2$$

can be looped to give a map

$$\Omega(FCY_1^1) \wedge \Omega(FCY_2^1) \to \Omega^2(FCY_3^2)$$

and that there is a commutative diagram

$$\begin{array}{ccc} FCY_1^0 \wedge FCY_2^0 & \longrightarrow & FCY_3^0 \\ {\scriptstyle \sigma_*\wedge\sigma_*}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \sigma_*} \\ \Omega(FCY_1^1) \wedge \Omega(FCY_2^1) & \longrightarrow & \Omega^2(FCY_3^2) \end{array} \qquad (7)$$

in which the maps σ_* (which are weak equivalences since the objects FCY_i are stably fibrant) are adjoint bonding maps.

It follows that the tensor product pairing

$$\mathcal{P}(X) \times \mathbf{M}(X) \xrightarrow{\otimes} \mathbf{M}(X)$$

induces cup product pairings

$$K_n(X) \otimes K'_m(X) \xrightarrow{\cup} K'_{n+m}(X)$$

for all n, m, which can (and in fact has been for some time) defined as the pairing in homotopy groups which is induced by the map

$$\Omega(K(\mathcal{P}(X))^1) \wedge \Omega(K(\mathbf{M}(X))^1) \xrightarrow{\cup} \Omega^2(K(\mathbf{M}(X))^2) \tag{8}$$

which, in turn, is induced by the pairing

$$s_\bullet(\mathcal{P}(X)) \times s_\bullet(\mathbf{M}(X)) \xrightarrow{\otimes} s^2_\bullet(\mathbf{M}(X)).$$

We have been writing $K(X)$ for "the" fibrant model of the symmetric spectrum $K(\mathcal{P}(X))$ and $K'(X)$ for "the" fibrant model of the symmetric spectrum $K(\mathbf{M}(X))$. We can and will write

$$K(X)^0 \wedge K'(X)^0 \xrightarrow{\cup} K'(X)^0$$

for the map (8).

Here are some applications of these ideas:

1) Suppose that $j : U \subset X$ is an open subscheme of X with reduced closed complement $Z = X - U$. The tensor product pairing

$$\mathcal{P}(X) \times \mathbf{M}(X) \xrightarrow{\otimes} \mathbf{M}(X)$$

respects restriction to open subsets U and preserves modules supported on $Z = X - U$, so that there is a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathbf{M}_{X-U} & \xrightarrow{\otimes} & \mathbf{M}_{X-U} \\ {\scriptstyle 1\times i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{P}(X) \times \mathbf{M}(X) & \xrightarrow{\otimes} & \mathbf{M}(X) \\ {\scriptstyle j^*\times j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathcal{P}(U) \times \mathbf{M}(U) & \xrightarrow[\otimes]{} & \mathbf{M}(U) \end{array} \qquad (9)$$

There is an induced biexact pairing

$$\mathcal{P}(X) \times \mathbf{M}(U) \xrightarrow{j^*\times 1} \mathcal{P}(U) \times \mathbf{M}(U) \xrightarrow{\otimes} \mathbf{M}(U)$$

so that the space $K(X)^0$ acts on the fibre sequence

$$K'_{X-U}(X)^0 \to K'(X)^0 \xrightarrow{j^*} K'(U)^0,$$

arising from the Localization Theorem (Theorem 16.1), where $K'_{X-U}(X)$ is the stably fibrant model for the (connective) symmetric spectrum $K(\mathbf{M}_{X-U})$ with homotopy groups

$$K'_{X-U}(X)_m = \pi^s_m K'_{X-U}(X) = \pi_m K'_{X-U}(X)^0.$$

It follows that there is a commutative diagram of cup product pairings

$$\begin{array}{ccc} K_n(X) \otimes K'_m(X) & \xrightarrow{\cup} & K'_{n+m}(X) \\ {\scriptstyle 1\otimes j^*}\downarrow & & \downarrow{\scriptstyle j_*} \\ K_n(X) \otimes K'_m(U) & \xrightarrow[\cup]{} & K'_{n+m}(U) \end{array}$$

and a corresponding induced pairing

$$K_n(X) \otimes K'_{X-U}(X)_m \xrightarrow{\cup} K'_{X-U}(X)_{n+m}.$$

One uses the relation (3) to show that there is a diagram

$$\begin{array}{ccc} K_n(X) \otimes K'_{m+1}(U) & \xrightarrow{\cup} & K'_{n+m+1}(U) \\ {\scriptstyle 1\otimes\partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ K_n(X) \otimes K'_{X-U}(X)_m & \xrightarrow[\cup]{} & K'_{X-U}(X)_{n+m} \end{array} \tag{10}$$

where ∂ is the boundary map in the long exact sequence which is associated to the fibre sequence

$$K'_{X-U}(X)^0 \to K'(X)^0 \xrightarrow{j^*} K'(U)^0.$$

2) Suppose that $\pi : Y \to X$ is a finite morphism of Noetherian schemes, and recall that such a map π induces a morphism $\pi_* : \mathbf{M}(Y) \to \mathbf{M}(X)$ in coherent sheaves (the transfer) and an inverse image map $\pi^* : \mathcal{P}(X) \to \mathcal{P}(Y)$ in vector bundles. There is a homotopy commutative diagram of biexact pairings

$$\begin{array}{cccc} & \mathcal{P}(Y) \times \mathbf{M}(Y) & \xrightarrow{\otimes} & \mathbf{M}(Y) \\ \mathcal{P}(X) \times \mathbf{M}(Y) \begin{array}{c} \nearrow^{\pi^*\times 1} \\ \searrow_{1\times\pi_*} \end{array} & & & \downarrow{\scriptstyle \pi_*} \\ & \mathcal{P}(X) \times \mathbf{M}(X) & \xrightarrow[\otimes]{} & \mathbf{M}(X) \end{array} \tag{11}$$

This diagram is homotopy commutative in the sense that it commutes up to a canonical morphism

$$P \otimes \pi_*(M) \to \pi_*(\pi^*(P) \otimes M)$$

which is an isomorphism for all vector bundles P on X and coherent sheaves M on Y. This diagram induces, in various levels of complexity, a homotopy commutative diagram

$$\begin{array}{ccccc} & & K(Y) \wedge K'(Y) & \xrightarrow{\cup} & K'(Y) \\ & \overset{\pi^* \wedge 1}{\nearrow} & & & \downarrow{\scriptstyle \pi_*} \\ K(X) \wedge K'(Y) & & & & \\ & \underset{1 \wedge \pi_*}{\searrow} & & & \\ & & K(X) \wedge K'(X) & \xrightarrow[\cup]{} & K'(X) \end{array} \tag{12}$$

of symmetric spectra, a homotopy commutative diagram

$$\begin{array}{ccccc} & & K(Y)^0 \wedge K'(Y)^0 & \xrightarrow{\cup} & K'(Y)^0 \\ & \overset{\pi^* \wedge 1}{\nearrow} & & & \downarrow{\scriptstyle \pi_*} \\ K(X)^0 \wedge K'(Y)^0 & & & & \\ & \underset{1 \wedge \pi_*}{\searrow} & & & \\ & & K(X)^0 \wedge K'(X)^0 & \xrightarrow[\cup]{} & K'(X)^0 \end{array} \tag{13}$$

of pointed spaces, and commutative diagrams of abelian group homomorphisms

$$\begin{array}{ccc} & \overset{\pi^*\otimes 1}{\nearrow} K_n(Y)\otimes K'_m(Y) \xrightarrow{\cup} & K'_{n+m}(Y) \\ K_n(X)\otimes K'_m(Y) & & \downarrow \pi_* \\ & \underset{1\otimes\pi_*}{\searrow} K_n(X)\otimes K'_m(X) \xrightarrow[\cup]{} & K'_{n+m}(X) \end{array} \tag{14}$$

In any of the forms (11), (12), (13) or (14), this phenomenon is called the *projection formula*.

3) Suppose again that U is an open subscheme of a Noetherian scheme X, and let $Z = X - U$ be the closed complement with the reduced subscheme structure. Recall that the category $\mathbf{M}(Z)$ can be identified up to equivalence with the subcategory $\mathbf{M}_I(X)$ of $\mathbf{M}_{X-U}$ which consists of those modules which are annihilated by the defining ideal I, and that this identification is induced by the transfer map $i_* : \mathbf{M}(Z) \to \mathbf{M}(X)$ which is associated to the closed immersion i. Recall further that the inclusion $\mathbf{M}_I(X) \to \mathbf{M}_{X-U}$ is a K-theory equivalence, by dévissage. The map i is a finite morphism of Noetherian schemes, so that there is

a projection formula

$$\begin{array}{ccccc} & \overset{i^*\times 1}{\nearrow} & \mathcal{P}(Z)\times\mathbf{M}(Z) & \xrightarrow{\otimes} & \mathbf{M}(Z) \\ \mathcal{P}(X)\times\mathbf{M}(Z) & & & & \downarrow i_* \\ & \underset{1\times i_*}{\searrow} & \mathcal{P}(X)\times\mathbf{M}(X) & \xrightarrow[\otimes]{} & \mathbf{M}(X) \end{array}$$

It follows that there is a homotopy commutative diagram of pairings

$$\begin{array}{ccccc} & \overset{i^*\times 1}{\nearrow} & \mathcal{P}(Z)\times\mathbf{M}(Z) & \xrightarrow{\otimes} & \mathbf{M}(Z) \\ \mathcal{P}(X)\times\mathbf{M}(Z) & & & & \downarrow i_* \\ & \underset{1\times i_*}{\searrow} & \mathcal{P}(X)\times\mathbf{M}_I(X) & \xrightarrow[\otimes]{} & \mathbf{M}_I(X) \end{array}$$

in which the maps i_* are K-theory equivalences.

It follows that in the diagram

$$\begin{array}{ccc} K(X)^0\wedge K'(Z)^0 & \longrightarrow & K'(Z)^0 \\ {\scriptstyle 1\wedge i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ K(X)^0\wedge K'(X)^0 & \xrightarrow{\cup} & K'(X)^0 \\ {\scriptstyle 1\wedge j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ K(X)^0\wedge K'(U)^0 & \xrightarrow[\cup]{} & K'(U)^0 \end{array}$$

the induced pairing $K(X)^0\wedge K'(Z)^0 \to K'(Z)^0$ on the homotopy fibre of j^* coincides up to homotopy with the composite

$$K(X)^0\wedge K'(Z)^0 \xrightarrow{i^*\wedge 1} K(Z)^0\wedge K'(Z)^0 \xrightarrow{\cup} K'(Z)^0,$$

where the indicated cup product arises from the tensor product pairing

$$\mathcal{P}(Z) \times \mathbf{M}(Z) \xrightarrow{\otimes} \mathbf{M}(Z).$$

We have proved the following:

Lemma 18.1. *Suppose that X is a Noetherian scheme, with open subscheme $i : U \subset X$ and (reduced) closed complement $j : Z \subset X$. Suppose that $v \in K_n(X)$ and $b \in K'_{m+1}(U)$. Then*

$$\partial(v \cup b) = i^*(v) \cup \partial(b)$$

in $K_{n+m}(Z)$.

Here are some other things to notice:

1) Tensor product is commutative up to natural isomorphism, meaning that the diagram of biexact pairings

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathbf{M}(X) & \xrightarrow{\otimes} & \mathbf{M}(X) \\ {\scriptstyle \tau}\downarrow{\scriptstyle \cong} & \nearrow_{\otimes} & \\ \mathbf{M}(X) \times \mathcal{P}(X) & & \end{array}$$

commutes up to canonical natural isomorphism, where τ is the isomorphism which reverses factors. Thus $K(X)$ acts on $K'(X)$ on both the right and the left, and the induced cup product pairings are related by the equations

$$u \cup v = (-1)^{mn} v \cup u$$

in $K'_*(X)$, for $u \in K_n(X)$ and $v \in K'_m(X)$. The sign comes from the fact that the map

$$S^n \wedge S^m \xrightarrow{c_{n,m}} S^m \wedge S^m$$

which is induced by the shuffle $c_{n,m} \in \Sigma_{m+n}$ which moves the first n letters past the last m letters, in order, has degree mn.

2) Tensor product gives $K(X)$ the structure of a ring spectrum, and gives the spectrum $K'(X)$ the structure of a module spectrum over the ring spectrum $K(X)$. Further, all morphisms of schemes $\pi : Y \to X$ induce homomorphisms of ring spectra $\pi^* : K(X) \to K(Y)$. Restriction of scalars along π^* gives $K'(Y)$ the structure of a module spectrum over $K(X)$. When $\pi : Y \to X$ is finite scheme morphism, then the transfer homomorphism $\pi_* : K'(Y) \to K'(X)$ is $K(X)$-linear — this is the content of the projection formula (12).

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