Lecture 008 (March 9, 2011)

19 K-theory with coefficients

Suppose that n is some positive number. There is a natural cofibre sequence

$$E \xrightarrow{\times n} E \xrightarrow{p} E/n$$

in the category of spectra (or symmetric spectra), where $\times n$ is multiplication by n, meaning the map in the stable category represented by the "composite"

$$E \xrightarrow{\Delta} \prod_{i=1}^{n} E \underset{\simeq}{\xleftarrow{c}} \bigvee_{i=1}^{n} E \xrightarrow{\nabla} E,$$

where Δ is the diagonal map, ∇ is the fold map, and c is the canonical stable equivalence relating finite wedges and finite products. This construction can be made natural in symmetric spectra E: it's not hard to see that the defining cofibre sequence for E/n is weakly equivalent to the sequence

$$E \wedge_{\Sigma} S \xrightarrow{1\wedge_{\Sigma}(\times n)} E \wedge_{\Sigma} S \xrightarrow{1\wedge_{\Sigma} p} E \wedge_{\Sigma} S/n$$

where S is the sphere spectrum. It follows that there is a stable equivalence

$$E/n \simeq E \wedge_{\Sigma} S/n$$

so that E/n may be constructed from E by smashing with the "Moore spectrum" S/n.

Alternatively, let $P(n)$ be the homotopy cofibre of the map $\times n : S^1 \to S^1$. Then the comparison of homotopy fibre sequences

$$\begin{array}{ccccc} \Omega(E/n) & \longrightarrow & E & \xrightarrow{\times n} & E \\ \vdots & & \downarrow \simeq & & \downarrow \simeq \\ \mathbf{hom}_*(P(n), E))[1] & \longrightarrow & \mathbf{hom}_*(S^1, E)[1] & \xrightarrow[(\times n)^*]{} & \mathbf{hom}_*(S^1, E)[1] \end{array}$$

implies that there is a stable equivalence

$$\Omega(E/n) \simeq \mathbf{hom}_*(P(n), E)[1],$$

(after making E fibrant, at least), so that there are stable equivalences

$$E/n \simeq \Omega(E/n)[1] \simeq \mathbf{hom}_*(P(n), E)[2]. \quad (1)$$

Any cofibre sequence in symmetric spectra is a fibre sequence, and so any cofibre sequence induces a fibre sequence in associated stably fibrant models, and hence induces a long exact sequence in stable homotopy groups for symmetric spectra, as defined above. It follows in particular that there is a natural long exact sequence

$$\dots \to \pi_{n+1}(E/n) \xrightarrow{\partial} \pi_n(E) \xrightarrow{\times n} \pi_n(E) \to \pi_n(E/n) \xrightarrow{\partial} \dots$$

and corresponding natural short exact sequences

$$0 \to \pi_n(E) \otimes \mathbb{Z}/n \to \pi_n(E/n) \to \mathrm{Tor}(\mathbb{Z}/n, \pi_{n-1}(E)) \to 0.$$

Thus, if E is connective then E/n is connective.

Smashing with a fixed symmetric spectrum preserves stable cofibre sequences (in a derived sense), and therefore also preserves stable fibre sequences. Thus if E acts on a fibre sequence $F_1 \to F_2 \to F_3$ in the sense that there is a commutative diagram

$$\begin{array}{ccccc} E \wedge_\Sigma F_1 & \longrightarrow & E \wedge_\Sigma F_2 & \longrightarrow & E \wedge_\Sigma F_3 \\ \downarrow & & \downarrow & & \downarrow \\ F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 \end{array}$$

then the induced comparison

$$\begin{array}{ccccc} E \wedge_\Sigma F_1 \wedge_\Sigma S/n & \longrightarrow & E \wedge_\Sigma F_2 \wedge_\Sigma S/n & \longrightarrow & E \wedge_\Sigma F_3 \wedge_\Sigma S/n \\ \downarrow & & \downarrow & & \downarrow \\ F_1 \wedge_\Sigma S/n & \longrightarrow & F_2 \wedge_\Sigma S/n & \longrightarrow & F_3 \wedge_\Sigma S/n \end{array}$$

gives a fibre sequence $F_1/n \to F_2/n \to F_3/n$ with an action by E.

Suppose that $\mathbf{M}$ is an exact category. Then $K(\mathbf{M})/n$ is the mod n K-theory spectrum, and it is standard to write

$$K_p(\mathbf{M}, \mathbb{Z}/n) = \pi_p(K(\mathbf{M})/n).$$

Thus, for a scheme (respectively Noetherian scheme) X we write

$$K_p(X, \mathbb{Z}/n) = \pi_p(K(X)/n)$$

for the mod n K-groups of X and

$$K'_p(X, \mathbb{Z}/n) = \pi_p(K'(X)/n)$$

for the mod n K'-groups of X.

Example: Bott elements

1) Suppose that k is a field with $char(k)$ not dividing n. Suppose that k contains a primitive n^{th} root of unity ξ.

The composite

$$Bk^* \to BGl(k) \to K(k)^0$$

(which is a π_1-isomorphism) induces a map of spectra

$$\Sigma^\infty Bk^* \to K(k)$$

The induced map

$$\begin{aligned}\pi_2\mathbf{hom}_*(P(n), \Sigma^\infty Bk^*)[2] \to & \pi_2\mathbf{hom}_*(P(n), K(k))[2] \\ & = K_2(k, \mathbb{Z}/n)\end{aligned}$$

coincides with the map

$$\pi_0\mathbf{hom}_*(P(n), \Sigma^\infty Bk^*) \to \pi_0\mathbf{hom}_*(P(n), K(k)).$$

Then the composite

$$\begin{array}{ccc} \pi_0\mathbf{hom}_*(P(n), Bk^*) & \longrightarrow & \pi_0\mathbf{hom}_*(P(n), \Sigma^\infty Bk^*) \\ & & \downarrow \\ & & \pi_0\mathbf{hom}_*(P(n), K(k)) \end{array}$$

defines a map $\phi : {}_nk^* \to K_2(k, \mathbb{Z}/n)$ which splits the canonical surjection

$$K_2(k, \mathbb{Z}/n) \twoheadrightarrow {}_nk^*. \tag{2}$$

An element $\beta \in K_2(k, \mathbb{Z}/n)$ which maps to ξ under the surjection (2) is called a *Bott element.* Write $\beta = \phi(\xi)$, and let this be a fixed choice of Bott element in all that follows (there are others, given by other primitive roots).

2) If k is algebraically closed, then $K_2(k)$ is uniquely divisible [3] so that

$$K_2(k, \mathbb{Z}/n) \cong \mu_n$$

(n^{th} roots of unity) with generator β. More generally, Suslin's rigidity theorem [4], [5] (and a comparison with KU/ℓ) implies that multiplication by the Bott element induces a map

$$K(k)/\ell(k) \xrightarrow{\beta} \Omega^2 K/\ell(k)$$

which induces an isomorphism in stable homotopy groups π_j for $j \geq 0$, so there are isomorphisms

$$K_{2k}(k, \mathbb{Z}/n) \cong \mu_n^{\otimes k},$$

for $k > 0$ and

$$K_{2r+1}(k, \mathbb{Z}/n) = 0$$

for all $r \geq 0$.

The bad news is that the mod n K-theory spectrum $K(X)/n$ may not have a ring spectrum structure in general, because the Moore spectrum S/n may not have a ring spectrum structure — see [6, A.6].

In all that follows, let ℓ be a prime which is distinct from the characteristic of k and let $n = \ell^\nu$, where $\nu \geq 2$ if $\ell = 3$ and $\nu \geq 4$ if $\ell = 2$ (these choices are made precisely so that S/n has a ring spectrum structure).

Subject to these conditions, the ring structure on $K_*(k, \mathbb{Z}/n)$ is defined by tensor product in the obvious way, and there is a ring isomorphism

$$\mathbb{Z}/n[\beta] \cong K_*(k, \mathbb{Z}/n).$$

3) Suppose now that k **does not** contain a primitive n^{th} root of unity ($n = \ell^\nu$), and let $\xi \in \overline{k}$ be a fixed choice of primitive root in the algebraic closure $\overline{k}$.

The field $k(\xi)$ is the splitting field for the polynomial $X^n - 1$. Let $f(x)$ be the irreducible polynomial for ξ (of degree d) and let G be the Galois group for $k(\xi)/k$. Then G acts on $\mu_n \subset k(\xi)^*$. If ζ is a root of the polynomial $f(X)$ (hence also a primitive n^{th} root of unity), then $\zeta \in K_2(k(\zeta), \mathbb{Z}/n)$ via the map

$$\phi : {}_n k(\zeta)^* \to K_2(k(\zeta), \mathbb{Z}/n)$$

described above. The product element

$$\beta_* = \prod_{f(\zeta)=0} \zeta \in K_{2d}(k(\xi), \mathbb{Z}/n)$$

is G-invariant, and is non-zero since the element $\beta^d \neq 0$ in $K_{2d}(\overline{k}, \mathbb{Z}/n)$, and β_* maps to a non-zero multiple of β^d.

Finally, consider the base change morphism

$$i^* : K_*(k, \mathbb{Z}/n) \to K_*(k(\xi), \mathbb{Z}/n)$$

as well as the transfer

$$i_* : K_*(k(\xi), \mathbb{Z}/n) \to K_*(k, \mathbb{Z}/n).$$

One can show at the exact category level (exercise) that the map

$$i_* i^* : K_*(k, \mathbb{Z}/n) \to K_*(k, \mathbb{Z}/n)$$

is multiplication by the degree d, and that the composite

$$i^* i_* : K_*(k(\xi), \mathbb{Z}/n) \to K_*(k(\xi), \mathbb{Z}/n)$$

is multiplication by the norm element $N = \sum_{g \in G} g$ in the evident Galois module structure. The inverse image i^* takes values in the G-invariants $K_*(k(\xi), \mathbb{Z}/n)^G$, and the $i^* i_*$ restricts to the map

$$K_*(k(\xi), \mathbb{Z}/n)^G \to K_*(k(\xi), \mathbb{Z}/n)^G$$

which is multiplication by the degree d. The element $[d]$ is a unit of $\mathbb{Z}/\ell^r$ for all r, since $d | (\ell^n - 1)$ so that ℓ does not divide d. It follows that transfer and base change define an isomorphism

$$K_*(k, \mathbb{Z}/n) \cong K_*(k(\xi), \mathbb{Z}/n)^G,$$

so that $\beta^d \in K_{2d}(k, \mathbb{Z}/n)$.

In other words, some power of the Bott element is always in the K-theory of the base field k, under the assumptions that we have made on the coefficients.

20 K-theory with finite coefficients, and homology

Suppose that ℓ is a prime number.

Lemma 20.1. *Suppose that X is a simply connected space. Then the homotopy groups of X are uniquely ℓ-divisible if and only if*

$$\tilde{H}_*(X, \mathbb{Z}/\ell) = 0$$

Here (and as usual), $\tilde{H}_*(X, \mathbb{Z}/\ell)$ is the reduced mod ℓ homology of X: it is the kernel of the map

$$H_*(X, \mathbb{Z}/\ell) \to H_*(*, \mathbb{Z}/\ell).$$

The requirement that $\tilde{H}_*(X, \mathbb{Z}/\ell) = 0$ is equivalent to saying that X has the mod ℓ homology of a point.

Proof. Suppose that A is a uniquely ℓ-divisible abelian group. Then $H_1(BA) \cong A$ is uniquely ℓ-divisible, so that $H_1(BA, \mathbb{Z}/\ell) = 0$. Suppose that $H_i(BA, \mathbb{Z}/\ell) = 0$ for $1 \leq i \leq r$. Then multiplication by ℓ on BA is given by a composite

$$BA \xrightarrow{\Delta} BA^{\times \ell} \xrightarrow{\nabla} BA$$

and this composite is an isomorphism of simplicial abelian groups. The induced map

$$H_{r+1}(BA, \mathbb{Z}/\ell) \xrightarrow{(\times \ell)_*} H_{r+1}(BA, \mathbb{Z}/\ell)$$

is multiplication by ℓ by a Künneth formula argument, and this composite is an isomorphism. The homology groups $\tilde{H}_*(X, \mathbb{Z}/\ell)$ for any space X are ℓ^2-torsion, so it follows that $H_{r+1}(BA, \mathbb{Z}/\ell) = 0$. Thus, inductively, $\tilde{H}_*(BA, \mathbb{Z}/\ell) = 0$. It follows from a standard Serre spectral sequence argument that

$$\tilde{H}_*(K(A, n), \mathbb{Z}/\ell) = 0$$

for all $n \geq 1$.

Suppose that X is a simply connected space with uniquely ℓ-divisible homotopy groups. The Postnikov sections P_nX have the same property, and there are fibre sequences

$$K(\pi_n(X), n) \to P_nX \to P_{n-1}X.$$

We know that $\tilde{H}_*(K(\pi_n(X), n), \mathbb{Z}/\ell) = 0$. Thus an inductive Serre spectral sequence argument shows that $\tilde{H}_*(P_n(X), \mathbb{Z}/\ell) = 0$ for all $n \geq 2$. A Serre spectral sequence argument also shows that the map $\pi : X \to P_nX$ induces isomorphisms

$$\tilde{H}_k(X, \mathbb{Z}/\ell) \cong \tilde{H}_k(P_n(X), \mathbb{Z}/\ell) = 0$$

for $0 \leq k \leq n$. By taking n sufficiently large, we see that

$$\tilde{H}_k(X, \mathbb{Z}/\ell) = 0$$

for all $k \geq 0$.

Suppose, conversely, that $\tilde{H}_*(X, \mathbb{Z}/\ell) = 0$, and let $\pi_k(X)$ be the bottom non-vanishing homotopy group. Then $\pi_k(X) = H_k(X)$ by the Hurewicz Theorem, and $H_k(X)$ is uniquely ℓ-divisible since $\tilde{H}_*(X, \mathbb{Z}/\ell) = 0$. There is a fibre sequence

$$F \to X \to K(\pi_k(X), k)$$

where F is k-connected and $k \geq 2$. Then

$$\tilde{H}_*(K(\pi_k(X), k), \mathbb{Z}/\ell) = 0$$

by the first paragraph, so a Serre spectral sequence argument shows that $\tilde{H}_*(F, \mathbb{Z}/\ell) = 0$. Then

$$\pi_{k+1}(F) \cong \pi_{k+1}(X)$$

is uniquely ℓ-divisible. Inductively, all homotopy groups of X are uniquely ℓ-divisible. □

Recall that the fundamental groupoid $\pi(Y)$, for a Kan complex Y, can be constructed to have the vertices of Y as objects and naive homotopy classes of paths $\Delta^1 \to Y$ rel. end points as morphisms. The composition laws

$$\pi(Y)(x, y) \times \pi(Y)(y, z) \to \pi(Y)(x, z)$$

are defined by 2-simplex fill-ins. There is a canonical map $\pi : Y \to B(\pi(Y))$ which is the identity on vertices, and takes an n-simplex $\sigma : \Delta^n \to X$ to the string of morphisms

$$\sigma(0) \to \sigma(1) \to \cdots \to \sigma(n)$$

which are determined by the non-degenerate faces

$$\Delta^1 \subset \Delta^n \xrightarrow{\sigma} Y$$

of σ. The induced group homomorphisms

$$\pi_1(Y, x) \to \pi(Y)(x, x)$$

are isomorphisms, by construction. If Z is another choice of Kan complex, then the groupoid homomorphism

$$\pi(Y \times Z) \to \pi(Y) \times \pi(Z)$$

is an isomorphism. Any homotopy $Y \times \Delta^1 \to Z$ induces a homotopy

$$h_* : \pi(Y) \times \mathbf{1} \to \pi(Z),$$

which is defined, as a natural transformation, by the images of the 1-simplices

$$\Delta^1 \xrightarrow{(y,1)} Y \times \Delta^1 \xrightarrow{h} Z.$$

Now suppose that X is a connected H-space. We can suppose that X is a Kan complex — see the arguments in Section 12 (Lecture 005).

It follows that from the paragraph above that the space $B(\pi(X))$ is a connected H-space, and that the canonical map $\pi : X \to B(\pi(X))$ is multiplicative. The map π is also surjective (actually an isomorphism) on fundamental groups, so that as in the proof of Lemma 12.1 the fundamental groupoid of $B(\pi(X))$ (which is $\pi(X)$) acts trivially on the homology $H_*(F, \mathbb{Z}/\ell)$ of the homotopy fibres F of the map π, and so the corresponding Serre spectral sequence has the standard form

$$\begin{aligned} E_2^{p,q} = H_p(B(\pi(X)), &H_q(F, \mathbb{Z}/\ell)) \\ &\Rightarrow H_{p+q}(X, \mathbb{Z}/\ell). \end{aligned} \tag{3}$$

Recall that the homotopy fibre F of the map $\pi : X \to B(\pi(X))$ is the universal cover of X.

Lemma 20.2. *Suppose that X is a connected H-space. Then the homotopy groups of X are uniquely ℓ-divisible if and only if $\tilde{H}_*(X, \mathbb{Z}/\ell) = 0$.*

Proof. Suppose that all of the homotopy groups of X are uniquely ℓ-divisible. Then the homotopy groups of the universal cover F of X are uniquely ℓ-divisible, so that $\tilde{H}_*(F, \mathbb{Z}/\ell) = 0$ by Lemma 20.1. An argument using the spectral sequence

(3) then shows that the map

$$H_*(X, \mathbb{Z}/\ell) \to H_*(B(\pi(X)), \mathbb{Z}/\ell)$$

is an isomorphism. But then there is a weak equivalence $B(\pi(X)) \simeq B(\pi_1(X))$ and $\pi_1(X)$ is a uniquely ℓ-divisible abelian group so that $\tilde{H}_*(B(\pi_1(X)), \mathbb{Z}/\ell) = 0$ by the proof of Lemma 20.1.

Suppose conversely that $\tilde{H}_*(X, \mathbb{Z}/\ell) = 0$. Then $\pi_1(X) \cong H_1(X)$ is a uniquely ℓ-divisible abelian group, so that

$$\tilde{H}_*(B(\pi(X)), \mathbb{Z}/\ell) = 0.$$

From the spectral sequence (3), we then conclude that $\tilde{H}_*(F, \mathbb{Z}/\ell) = 0$. But this means that all of the homotopy groups of the universal cover F of X are uniquely ℓ-divisible, by Lemma 20.1. Thus, all of the homotopy groups of X are uniquely ℓ-divisible. □

Lemma 20.3. *Suppose that $f : A \to A'$ is a homomorphism of abelian groups. Then f induces an isomorphism*

$$f_* : H_*(BA, \mathbb{Z}/\ell) \xrightarrow{\cong} H_*(BA', \mathbb{Z}/\ell)$$

if and only if the groups $\ker(f)$ *and* $\operatorname{cok}(f)$ *are uniquely ℓ-divisible.*

Proof. If $\ker(f)$ and $\operatorname{cok}(f)$ are uniquely ℓ-divisible, then the maps $BA \twoheadrightarrow B\operatorname{im}(f)$ and $B\operatorname{im}(f) \hookrightarrow BA'$ induce $\tilde{H}_*(\ ,\mathbb{Z}/\ell)$-isomorphisms, by Serre spectral sequence arguments, since

$$\tilde{H}_*(B\ker(f),\mathbb{Z}/\ell) \cong \tilde{H}_*(B\operatorname{cok}(f),\mathbb{Z}/\ell) = 0.$$

Conversely, if the map

$$f_* : H_*(BA,\mathbb{Z}/\ell) \to H_*(BA',\mathbb{Z}/\ell)$$

is an isomorphism, then the map

$$f_* : H_*(B^3A,\mathbb{Z}/\ell) \to H_*(B^3A',\mathbb{Z}/\ell)$$

is an isomorphism, by an iterated Künneth formula argument. The homotopy fibre F of the map $f_* : B^3A \to B^3A'$ is a simply connected space with $\tilde{H}_*(F,\mathbb{Z}/\ell) = 0$. Lemma 20.1 implies that the homotopy groups of F are uniquely ℓ-divisible. The non-trivial homotopy groups of F are $\pi_3(F) = \ker(f)$ and $\pi_2(F) = \operatorname{cok}(f)$, so that $\ker(f)$ and $\operatorname{cok}(f)$ are uniquely ℓ-divisible. □

Theorem 20.4. *Suppose that $f : \mathcal{O} \to \mathcal{O}'$ is a local homomorphism of local rings. Then the induced map*

$$f_* : K(\mathcal{O})/\ell \to K(\mathcal{O}')/\ell$$

is a stable equivalence if and only if the map

$$f_* : H_*(BGl(\mathcal{O}), \mathbb{Z}/\ell) \to H_*(BGl(\mathcal{O}'), \mathbb{Z}/\ell)$$

is an isomorphism.

The requirement that $f : \mathcal{O} \to \mathcal{O}'$ is a local homomorphism means that $f(\mathcal{M}) \subset \mathcal{M}'$, where $\mathcal{M}$ and $\mathcal{M}'$ are the respective maximal ideals. Morphisms of this type include all residue maps $\mathcal{O} \to \mathcal{O}/\mathcal{M}$, all morphisms $k \to \mathcal{O}$ where k is a field, and all field homomorphisms $k \to L$.

Proof. Let F be the homotopy fibre of the map $K(\mathcal{O}) \to K(\mathcal{O}')$ in the stable category. Then F is a connective spectrum. The map

$$K(\mathcal{O})^0 \to K(\mathcal{O}')^0$$

is an isomorphism in π_0 (since $K_0(\mathcal{O}) \cong K_0(\mathcal{O}') \cong \mathbb{Z}$), so there is a fibre sequence

$$F^0 \to BGl(\mathcal{O})^+ \xrightarrow{f_*} BGl(\mathcal{O}')^+$$

by the $Q = +$ theorem (Theorem 11.1).

The standard inclusion $R^* \to Gl(R)$ of groups (R^* is units in R) induces a natural splitting

$$BR^* \to BGl(R) \xrightarrow{det} BR^*$$

of the determinant homomorphism. Thus, if the natural homomorphism $K_1(R) \to R^*$ is an isomorphism, then the long exact sequence for resulting fibre sequence

$$BSl(R)^+ \to BGl(R)^+ \to BR^*$$

breaks up into short exact sequences

$$0 \to \pi_n(BSl(R)^+) \to \pi_n(BGl(R)^+) \to \pi_n(BR^*) \to 0$$

which are split by the induced map $\pi_n(BR^*) \to \pi_n(BGl(R)^+)$. It follows that the composite

$$BSl(R)^+ \times BR^* \to BGl(R)^+ \times BGl(R)^+ \xrightarrow{\oplus} BGl(R)^+$$

is a weak equivalence in all such cases.

It follows that there is a homotopy commutative diagram

$$\begin{array}{ccc} BSl(\mathcal{O})^+ \times B\mathcal{O}^* & \longrightarrow & BSl(\mathcal{O}')^+ \times B\mathcal{O}'^* \\ \simeq \downarrow & & \downarrow \simeq \\ BGl(\mathcal{O})^+ & \xrightarrow[f_*]{} & BGl(\mathcal{O}')^+ \end{array}$$

in which the vertical maps are weak equivalences. It also follows that the map $f_* : BGl(\mathcal{O}) \to BGl(\mathcal{O}')$ is an $H_*(\, , \mathbb{Z}/\ell)$-isomorphism if and only if the maps

$$BSl(\mathcal{O})^+ \xrightarrow{f_*} BSl(\mathcal{O}')^+$$

and

$$B\mathcal{O}^* \to B\mathcal{O}'^*$$

are $H_*(\ ,\mathbb{Z}/\ell)$-isomorphisms.

The map $BSl(\mathcal{O})^+ \to BSl(\mathcal{O}')^+$ is the map of spaces in level 0 of the 1-connected covers

$$K(\mathcal{O})(1) \to K(\mathcal{O}')(1)$$

of the respective K-theory spectra. Let E be the homotopy fibre of this map. Then the space E^0 is a connected H-space, and so by Lemma 20.2 the homotopy groups of E^0 are uniquely ℓ-divisible if and only if $\tilde{H}_*(E^0, \mathbb{Z}/\ell) = 0$. But this is true if and only if the map

$$f_* : BSl(\mathcal{O})^+ \to BSl(\mathcal{O}')^+$$

is an $H_*(\ ,\mathbb{Z}/\ell)$-isomorphism, by a Serre spectral sequence argument.

It follows that the map

$$BGl(\mathcal{O}) \xrightarrow{f_*} BGl(\mathcal{O}')$$

is an $H_*(\ ,\mathbb{Z}/\ell)$-isomorphism if and only if the groups $\pi_*(E)$, and the kernel and cokernel of the map $f_* : \mathcal{O}^* \to \mathcal{O}'^*$ are all uniquely ℓ-divisible. The homotopy groups of the fibre F^0 of the map

$K(\mathcal{O})^0 \to K(\mathcal{O}')^0$ are of the form

$$\pi_j(F^0) = \begin{cases} \mathrm{cok}(\mathcal{O}^* \to \mathcal{O}'^*) & \text{if } j = 0, \\ \pi_1(E^0) \oplus \ker(\mathcal{O}^* \to \mathcal{O}'^*) & \text{if } j = 1, \text{ and} \\ \pi_j(E^0) & \text{if } j > 1. \end{cases}$$

It follows that the homotopy groups of F^0 are all uniquely ℓ-divisible if and only if the map

$$f_* : BGl(\mathcal{O}) \to BGl(\mathcal{O}')$$

is an $H_*(\ ,\mathbb{Z}/\ell)$-isomorphism. But this means that f_* is an $H_*(\ ,\mathbb{Z}/\ell)$-isomorphism if and only if the map

$$f_* : K(\mathcal{O})/\ell \to K(\mathcal{O}')/\ell$$

is a stable equivalence, because F/ℓ is stably trivial if and only if the homotopy groups of F^0, are uniquely ℓ-divisible. □

The n-connected cover $E(n)$ of a spectrum E is the fibre of the n^{th} Postnikov section $E \to P_n(E)$. A construction of the functor $E \mapsto P_n(E)$ is given in [2, Sec. 4.7], but the construction can also be fudged by playing with the diagrams

$$\begin{array}{ccccc} P_{n+k}E^k & \xrightarrow{P_{n+k}\sigma_*} & P_{n+k}\Omega E^{k+1} & \xrightarrow{P_{n+k}\Omega\pi_{n+k+1}} & P_{n+k}\Omega P_{n+k+1}E^{k+1} \\ & & \uparrow{\scriptstyle \pi_{n+k}} & & \uparrow{\scriptstyle \pi_{n+k}}\ \simeq \\ & & \Omega E^{k+1} & \xrightarrow[\Omega\pi_{n+k+1}]{} & \Omega P_{n+k+1}E^{k+1} \end{array}$$

Here, $\pi_r : Z \to P_r Z$ denotes the standard map taking values in the Postnikov section $P_r Z$ for a simplicial set Z.

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