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1 Simplicial sets

The finite ordinal number \mathbf{n} is the set of counting numbers

$$\mathbf{n} = \{0, 1, \dots, n\}.$$

There is an obvious ordering on this set which gives it the structure of a poset, and hence a (tiny) category.

Fact: If C is a category then the functors $\alpha : \mathbf{n} \rightarrow C$ can be identified with strings of arrows

$$\alpha(0) \rightarrow \alpha(1) \rightarrow \dots \rightarrow \alpha(n)$$

of length n .

The collection of all finite ordinal numbers and all order-preserving functions between them (aka. poset morphisms, or functors) form the **ordinal number** category Δ .

Examples:

1) The monomorphisms $d^i : \mathbf{n} - \mathbf{1} \rightarrow \mathbf{n}$ are defined by the strings of relations

$$0 \leq 1 \leq \cdots \leq i-1 \leq i+1 \leq \cdots \leq n$$

for $0 \leq i \leq n$. These morphisms are called **cofaces**.

2) The epimorphisms $s^j : \mathbf{n} + \mathbf{1} \rightarrow \mathbf{n}$ are defined by the strings

$$0 \leq 1 \leq \cdots \leq j \leq j \leq \cdots \leq n$$

for $0 \leq j \leq n$. These are the **codegeneracies**.

The cofaces and codegeneracies together satisfy the **cosimplicial identities**:

$$d^j d^i = d^i d^{j-1} \text{ if } i < j,$$

$$s^j s^i = s^i s^{j+1} \text{ if } i \leq j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i = j, j+1, \\ d^{i-1} s^j & \text{if } i > j+1. \end{cases}$$

Δ is generated by the cofaces and codegeneracies, subject to the cosimplicial identities [5].

Every morphism of Δ has a unique epi-monic factorization, and has a canonical form defined by strings of codegeneracies and strings of cofaces.

A **simplicial set** is a functor

$$X : \Delta^{op} \rightarrow \mathbf{Set},$$

ie. a contravariant set-valued functor on Δ .

We usually write $\mathbf{n} \mapsto X_n$, and X_n is called the set of **n -simplices** of X .

A **simplicial map** $f : X \rightarrow Y$ is a natural transformation of such functors.

The simplicial sets and simplicial maps form the **category of simplicial sets**, which is denoted by **$s\mathbf{Set}$** in these notes.

A simplicial set is a simplicial object in the set category. Generally, $s\mathcal{A}$ denotes the category of simplicial objects $\Delta^{op} \rightarrow \mathcal{A}$ in a category \mathcal{A} .

Examples include the categories **$s\mathbf{Gr}$** of simplicial groups, **$s(R - \mathbf{Mod})$** of simplicial R -modules, **$s(s\mathbf{Set}) = s^2\mathbf{Set}$** of bisimplicial sets, and so on.

Examples of simplicial sets:

1) The **topological standard n -simplex** is the space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum t_i = 1\}$$

The assignment $\mathbf{n} \mapsto |\Delta^n|$ defines a functor $\Delta \rightarrow \mathbf{Top}$, which is a **cosimplicial space**.

All spaces $|\Delta^n|$ are compact Hausdorff spaces, and it's better [1], [2] to assume that the topological standard n -simplices are compactly generated Hausdorff spaces. **CGHaus** is the category of such things.

If X is a space, then the **singular set** (or singular complex) $S(X)$ is defined by

$$S(X)_n = \text{hom}(|\Delta^n|, X).$$

The assignment $X \mapsto S(X)$ defines a functor

$$S : \mathbf{CGHaus} \rightarrow s\mathbf{Set},$$

called the **singular functor**.

2) The ordinal number \mathbf{n} represents a simplicial set

$$\Delta^n = \text{hom}_\Delta(\mathbf{\cdot}, \mathbf{n}),$$

which is called the **standard n -simplex**. Write

$$l_n \in \text{hom}_\Delta(\mathbf{n}, \mathbf{n}) = \Delta_n^n$$

for the identity map $1_{\mathbf{n}} : \mathbf{n} \rightarrow \mathbf{n}$.

ι_n is the **classifying n -simplex**: there is a natural bijection

$$\text{hom}_{\mathbf{sSet}}(\Delta^n, Y) \cong Y_n,$$

such that $\sigma : \Delta^n \rightarrow Y$ maps to $\sigma(\iota_n) \in Y_n$.

I say that a map $\Delta^n \rightarrow Y$ is an n -simplex of Y .

If $\sigma : \Delta^n \rightarrow X$ is a simplex of X , then the i^{th} **face** $d_i(\sigma)$ is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

while the j^{th} **degeneracy** $s_j(\sigma)$ is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

3) $\partial\Delta^n$ is the subobject of Δ^n which is generated by the $(n-1)$ -simplices d^i , $0 \leq i \leq n$, and let Λ_k^n be the subobject of $\partial\Delta^n$ which is generated by the simplices d^i , $i \neq k$. $\partial\Delta^n$ is called the **boundary** of Δ^n , and Λ_k^n is called the k^{th} **horn**.

The faces $d^i : \Delta^{n-1} \rightarrow \Delta^n$ determine a *covering*

$$\bigsqcup_{i=0}^n \Delta^{n-1} \rightarrow \partial\Delta^n,$$

and for each $i < j$ there are pullback diagrams

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{d^{j-1}} & \Delta^{n-1} \\ d^i \downarrow & & \downarrow d^i \\ \Delta^{n-1} & \xrightarrow{d^j} & \Delta^n \end{array}$$

It follows that there is a coequalizer

$$\bigsqcup_{i < j, 0 \leq i, j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{0 \leq i \leq n} \Delta^{n-1} \longrightarrow \partial \Delta^n$$

in $s\mathbf{Set}$. Similarly, there is a coequalizer

$$\bigsqcup_{i < j, i, j \neq k} \Delta^{n-2} \rightrightarrows \bigsqcup_{0 \leq i \leq n, i \neq k} \Delta^{n-1} \longrightarrow \Lambda_k^n.$$

4) Suppose that a category C is **small** in the sense that the morphisms $\text{Mor}(C)$ and objects $\text{Ob}(C)$ are sets.

Examples include all finite ordinal numbers \mathbf{n} , all monoids (small categories having one object), and all groups.

If C is a small category there is a simplicial set BC with

$$BC_n = \text{hom}(\mathbf{n}, C),$$

meaning the functors $\mathbf{n} \rightarrow C$. The simplicial structure on BC is defined by precomposition with ordinal number maps.

The object BC is called, variously, the **classifying space** or **nerve** of C . One often sees the notation NC for BC .

Note that $B\mathbf{n} = \Delta^n$ in this notation.

5) Suppose that I is a small category, and that $X : I \rightarrow \mathbf{Set}$ is a set-valued functor. The **category of elements** (or **translation category**, or **slice category**)

$$*/X = E_I(X)$$

associated to X has as objects all pairs (i, x) with $x \in X(i)$, or equivalently all functions

$$* \xrightarrow{x} X(i).$$

A morphism $\alpha : (i, x) \rightarrow (j, y)$ is a morphism $\alpha : i \rightarrow j$ of I such that $\alpha_*(x) = y$, or equivalently a commutative diagram

$$\begin{array}{ccc} & X(i) & \\ & \nearrow x & \\ * & & \\ & \searrow y & \\ & X(j) & \\ & \downarrow \alpha_* & \end{array}$$

The simplicial set $B(E_I X)$ is often called the **homotopy colimit** for the functor X , and one writes

$$\underline{\operatorname{holim}}_I X = B(E_I X).$$

Example: $BI = \underline{\operatorname{holim}}_I *$.

There is a canonical functor $E_I X \rightarrow I$ that is defined by the assignment $(i, x) \mapsto i$. It induces a canonical simplicial set map

$$\pi : B(E_I X) = \underline{\text{holim}}_I X \rightarrow BI.$$

The functors $\mathbf{n} \rightarrow E_I X$ can be identified with strings

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n).$$

Such a string is uniquely specified by the underlying string $i_0 \rightarrow \dots \rightarrow i_n$ in the index category I and $x_0 \in X(i_0)$. It follows that there is an identification

$$(\underline{\text{holim}}_I X)_n = B(E_I X)_n = \bigsqcup_{i_0 \rightarrow \dots \rightarrow i_n} X(i_0).$$

The construction is natural with respect to natural transformations in X . Thus a diagram $X : I \rightarrow \mathbf{sSet}$ in simplicial sets determines a bisimplicial set with (n, m) simplices

$$B(E_I X)_m = \bigsqcup_{i_0 \rightarrow \dots \rightarrow i_n} X(i_0)_m.$$

The **diagonal** $d(Y)$ of a bisimplicial set Y is the simplicial set with n -simplices $Y_{n,n}$. Equivalently, $d(Y)$ is the composite functor

$$\Delta^{op} \xrightarrow{\Delta} \Delta^{op} \times \Delta^{op} \xrightarrow{Y} \mathbf{Set}$$

where Δ is the diagonal functor.

The diagonal $dB(E_I X)$ of the bisimplicial set $B(E_I X)$ is the **homotopy colimit** $\underline{\text{holim}}_I X$ in $s\mathbf{Set}$ of the functor $X : I \rightarrow s\mathbf{Set}$, and there is a natural simplicial set map

$$\pi : \underline{\text{holim}}_I X \rightarrow BI.$$

6) Suppose that X and Y are simplicial sets. There is a simplicial set $\mathbf{hom}(X, Y)$ with n -simplices

$$\mathbf{hom}(X, Y)_n = \text{hom}(X \times \Delta^n, Y),$$

called the **function complex**.

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X, Y) \rightarrow Y$$

which sends the pair $(x, f : X \times \Delta^n \rightarrow Y)$ to the simplex $f(x, \iota_n)$.

Suppose that K is another simplicial set. The function

$$ev_* : \text{hom}(K, \mathbf{hom}(X, Y)) \rightarrow \text{hom}(X \times K, Y),$$

which is defined by sending the map $g : K \rightarrow \mathbf{hom}(X, Y)$ to the composite

$$X \times K \xrightarrow{1 \times g} X \times \mathbf{hom}(X, Y) \xrightarrow{ev} Y,$$

is a natural bijection, giving the **exponential law**

$$\text{hom}(K, \mathbf{hom}(X, Y)) \cong \text{hom}(X \times K, Y).$$

This natural isomorphism gives $s\mathbf{Set}$ the structure of a cartesian closed category.

The function complexes also give $s\mathbf{Set}$ the structure of a category enriched in simplicial sets.

2 The simplex category and realization

The **simplex category** Δ/X for a simplicial set X has for objects all simplices $\Delta^n \rightarrow X$; its morphisms are the incidence relations between the simplices, meaning all commutative diagrams

$$\begin{array}{ccc} \Delta^m & & \\ \theta \downarrow & \searrow \tau & \\ \Delta^n & \nearrow \sigma & X \end{array} \quad (1)$$

Fact: Every simplicial set X is a colimit of its simplices. The simplices $\Delta^n \rightarrow X$ define an isomorphism

$$\varinjlim_{\Delta^n \rightarrow X} \Delta^n \xrightarrow{\cong} X.$$

The **realization** $|X|$ of a simplicial set X is defined by

$$|X| = \varinjlim_{\Delta^n \rightarrow X} |\Delta^n|.$$

The assignment $X \mapsto |X|$ defines a functor

$$|\cdot| : s\mathbf{Set} \rightarrow \mathbf{CGHaus}.$$

Lemma 2.1. *The realization functor is left adjoint to the singular functor $S : \mathbf{CGHaus} \rightarrow s\mathbf{Set}$.*

Proof. Exercise. □

Examples:

- 1) $|\Delta^n| = |\Delta^n|$, since the simplex category Δ/Δ^n has a terminal object, namely $1 : \Delta^n \rightarrow \Delta^n$.
- 2) $|\partial\Delta^n| = \partial|\Delta^n|$, and $|\Lambda_k^n|$ is the part of the boundary $\partial|\Delta^n|$ with the face opposite the vertex k removed, since the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The n^{th} **skeleton** $\text{sk}_n X$ of a simplicial set X is the subobject generated by the simplices X_i , $0 \leq i \leq n$.

The ascending sequence of subcomplexes

$$\text{sk}_0 X \subset \text{sk}_1 X \subset \text{sk}_2 X \subset \dots$$

defines a filtration of X , and there are pushouts

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} \partial \Delta^n & \longrightarrow & \text{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in NX_n} \Delta^n & \longrightarrow & \text{sk}_n X \end{array}$$

NX_n is the set of non-degenerate n -simplices of X .

Facts: 1) The realization of a simplicial set is a CW-complex.

2) Every monomorphism $A \rightarrow B$ of simplicial sets induces a cofibration $|A| \rightarrow |B|$ of spaces. In fact, $|B|$ is constructed from $|A|$ by attaching cells.

The following is more serious:

Lemma 2.2. *The realization functor preserves finite limits. Equivalently, it preserves finite products and equalizers.*

See [2] or [1] for a proof.

3 Model structure for simplicial sets

This section summarizes material which is presented in some detail in [4]. See also [2].

Say that a map $f : X \rightarrow Y$ of simplicial sets is a **weak equivalence** if the induced map $f_* : |X| \rightarrow |Y|$ is a weak equivalence of **CGHaus**.

A map $i : A \rightarrow B$ of simplicial sets is a **cofibration** if and only if it is a monomorphism, meaning that all functions $i : A_n \rightarrow B_n$ are injective.

A simplicial set map $p : X \rightarrow Y$ is a **fibration** if and only if it has the right lifting property with respect to all trivial cofibrations.

Theorem 3.1. *With these definitions of weak equivalence, cofibration and fibration, the category $s\mathbf{Set}$ of simplicial sets satisfies the axioms for a closed model category.*

Here are the basic ingredients of the proof:

Lemma 3.2. *A map $p : X \rightarrow Y$ is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

The proof of this Lemma is formal.

The following can be proved with simplicial approximation techniques [3]:

Lemma 3.3. *Suppose that a simplicial set X has at most countably many non-degenerate simplices.*

Then the set of path components $\pi_0|X|$ and all homotopy groups $\pi_n(|X|, x)$ are countable.

Here's a consequence:

Lemma 3.4 (Bounded cofibration lemma). *Suppose that $i : X \rightarrow Y$ is a trivial cofibration and that $A \subset Y$ is a countable subcomplex.*

Then there is a countable subcomplex $B \subset Y$ with $A \subset B$ such that the map $B \cap X \rightarrow B$ is a trivial cofibration.

Lemma 3.4 means that the set of countable trivial cofibrations generate the class of all trivial cofibrations, while the set of all inclusions $\partial\Delta^n \subset \Delta^n$ generates the class of all cofibrations.

Theorem 3.1 then follows from small object arguments.

Remark 3.5. The realization functor preserves cofibrations and trivial cofibrations, so the singular functor S preserves fibrations and trivial fibrations, by adjointness. The adjoint pair

$$| | : s\mathbf{Set} \rightleftarrows \mathbf{CGHaus} : S,$$

is a **Quillen adjunction**.

Kan fibrations

A **Kan fibration** is a map $p : X \rightarrow Y$ of simplicial sets which has the right lifting property with respect to all inclusions $\Lambda_k^n \subset \Delta^n$.

A **Kan complex** is a simplicial set X for which the canonical map $X \rightarrow *$ is a Kan fibration.

Every fibration is a Kan fibration. Every fibrant simplicial set is a Kan complex.

Kan complexes Y have combinatorially defined homotopy groups: if $x \in Y_0$ is a vertex of Y , then

$$\pi_n(Y, x) = \pi((\Delta^n, \partial\Delta^n), (Y, x))$$

where $\pi(,)$ denotes simplicial homotopy classes of maps.

The set of **path components** $\pi_0 X$ of a simplicial set X is defined by the coequalizer

$$X_1 \rightrightarrows X_0 \rightarrow \pi_0 X,$$

where the maps $X_1 \rightarrow X_0$ are the face maps d_0, d_1 .

Say that a map $f : Y \rightarrow Y'$ of Kan complexes is a **combinatorial weak equivalence** if it induces isomorphisms

$$\pi_n(Y, x) \xrightarrow{\cong} \pi_n(Y', f(x))$$

for all $x \in Y_0$, and

$$\pi_0(Y) \xrightarrow{\cong} \pi_0(Y').$$

Going further requires the following major theorem, due to Quillen:

Theorem 3.6. *The realization of a Kan fibration is a Serre fibration.*

The proof of this result uses much of the classical homotopy theory of Kan complexes (in particular the theory of minimal fibrations), and will not be discussed here.

Here are the consequences:

Theorem 3.7 (Milnor theorem). *Suppose that Y is a Kan complex and that $\eta : Y \rightarrow S(|Y|)$ is the adjunction homomorphism. Then η is a combinatorial weak equivalence.*

It follows that the combinatorial homotopy groups of $\pi_n(Y, x)$ coincide up to natural isomorphism with the ordinary homotopy groups $\pi_n(|Y|, x)$ of the realization, for all Kan complexes Y .

The proof of Theorem 3.7 is a long exact sequence argument using path-loop fibre sequences in simplicial sets. These are Kan fibre sequences, and

the key is to know that their realizations are fibre sequences.

Theorem 3.8. *Every Kan fibration is a fibration.*

The key step in the proof of Theorem 3.8 is to show, using Theorem 3.7, that every map which is a Kan fibration and a weak equivalence is a trivial fibration.

This is used to show that every trivial cofibration has the left lifting property with respect to all Kan fibrations, or that every Kan fibration has the right lifting property with respect to all trivial cofibrations.

Remark 3.9. Theorem 3.8 implies that the model structure of Theorem 3.1 consists of cofibrations, Kan fibrations and weak equivalences.

This is the standard, classical model structure for simplicial sets.

The identification of the fibrations with Kan fibrations is the “hard” part of its construction.

Theorem 3.10. *The adjunction maps $\eta : X \rightarrow S(|X|)$ and $\varepsilon : |S(Y)| \rightarrow Y$ are weak equivalences, for all simplicial sets X and spaces Y , respectively.*

In particular, the standard Quillen model structures on **sSet** and **CGHaus** are Quillen equivalent.

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