Contents

5 Grothendieck topologies

A **Grothendieck site** is a small category $\mathscr{C}$ equipped with a Grothendieck topology $\mathscr{T}$.

A **Grothendieck topology** $\mathscr{T}$ consists of a collection of subfunctors

$$R \subset \hom(\ ,U), \quad U \in \mathscr{C},$$

called **covering sieves**, such that the following hold:

1) (**base change**) If $R \subset \hom(\ ,U)$ is covering and $\phi : V \to U$ is a morphism of $\mathscr{C}$, then

$$\phi^{-1}(R) = \{\gamma : W \to V \mid \phi \cdot \gamma \in R\}$$

is covering for V.

2) (**local character**) Suppose $R, R' \subset \hom(\ ,U)$ and R is covering. If $\phi^{-1}(R')$ is covering for all $\phi : V \to U$ in R, then R' is covering.

3) $\hom(\ ,U)$ is covering for all $U \in \mathscr{C}$.

Typically, Grothendieck topologies arise from covering families in sites $\mathscr{C}$ having pullbacks. Covering families are sets of maps which generate covering sieves.

Suppose that $\mathscr{C}$ has pullbacks. A topology $\mathscr{T}$ on $\mathscr{C}$ consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \to U\}, \quad U \in \mathscr{C},$$

called **covering families**, such that

1) Suppose $\phi_\alpha : U_\alpha \to U$ is a covering family and $\psi : V \to U$ is a morphism of $\mathscr{C}$. Then the set of all $V \times_U U_\alpha \to V$ is a covering family for V.

2) Suppose $\{\phi_\alpha : U_\alpha \to V\}$ is covering, and $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha\}$ is covering for all α. Then the set of composites

$$W_{\alpha,\beta} \xrightarrow{\gamma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$

is covering.

3) The singleton set $\{1 : U \to U\}$ is covering for each $U \in \mathscr{C}$.

Examples:

1) $X =$ topological space. The site $\mathrm{op}|_X$ is the poset of open subsets $U \subset X$. A covering family for an open U is an open cover $V_\alpha \subset U$.

2) $X =$ topological space. The site $\mathrm{loc}|_X$ is the category of all maps $f : Y \to X$ which are local homeomorphisms.

$f : Y \to X$ is a **local homeomorphism** if each $x \in Y$ has a neighbourhood U such that $f(U)$ is open in X and the restricted map $U \to f(U)$ is a homeomorphism. A morphism of $\mathrm{loc}|_X$ is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

where f and f' are local homeomorphisms. A family $\{\phi_\alpha : Y_\alpha \to Y\}$ of local homeomorphisms (over X) is covering if $X = \cup \phi_\alpha(Y_\alpha)$.

3) $S =$ a scheme (topological space with sheaf of rings locally isomorphic to affine schemes $\mathrm{Sp}(R)$). The underlying topology on S is the Zariski topology.

The **Zariski site** $Zar|_S$ is the poset with objects all open subschemes $U \subset S$. A family $V_\alpha \subset U$ is covering if $\cup V_\alpha = U$ (as sets).

A scheme homomorphism $\phi : Y \to S$ is **étale** at $y \in Y$ if

a) $\mathscr{O}_y$ is a flat $\mathscr{O}_{f(y)}$-module (ϕ is flat at y).

b) ϕ is unramified at y: $\mathscr{O}_y / \mathscr{M}_{f(y)} \mathscr{O}_y$ is a finite separable field extension of $k(f(y))$.

Say that a map $\phi : Y \to S$ is **étale** if it is étale at every $y \in Y$ (and locally of finite type).

4) $S =$ scheme. The **étale site** $et|_S$ has as objects all étale maps $\phi : V \to S$ and all diagrams

$$\begin{array}{ccc} V & \longrightarrow & V' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & S & \end{array}$$

for morphisms (with ϕ, ϕ' étale).

An **étale cover** is a collection of étale morphisms $\phi_\alpha : V_\alpha \to V$ such that $V = \cup \phi_\alpha(V_\alpha)$.

Equivalently every morphism $\mathrm{Sp}(\Omega) \to V$ lifts to some V_α if Ω is a separably closed field.

5) The **Nisnevich site** $Nis|_S$ has the same underlying category as the étale site, namely all étale maps $V \to S$ and morphisms between them.

   A **Nisnevich cover** is a family of étale maps $V_\alpha \to V$ such that every morphism $\mathrm{Sp}(K) \to V$ lifts to some V_α where K is any field.

6) A **flat cover** of a scheme T is a set of flat morphisms $\phi_\alpha : T_\alpha \to T$ (ie. mophisms which are flat at each point) such that $T = \cup \phi_\alpha(T_\alpha)$ as a set (equivalently $\sqcup T_\alpha \to T$ is faithfully flat).

   $(Sch|_S)_{fl}$ is the "big" **flat site**.

Here's a trick: pick a large cardinal κ; then $(Sch|_S)$ is the category of S-schemes $X \to S$ such that the cardinality of both the underlying point set of X and all sections $\mathcal{O}_X(U)$ of its sheaf of rings are bounded above by κ.

7) There are corresponding big sites $(Sch|_S)_{Zar}$, $(Sch|_S)_{et}$, $(Sch|_S)_{Nis}$, ... and you can play similar games with topological spaces.

8) Suppose that $G = \{G_i\}$ is profinite group such that all $G_j \to G_i$ are surjective group homomorphisms. Write also $G = \varprojlim G_i$.

   A **discrete G-set** is a set X with G-action which factors through an action of G_i for some i.

   $G - \mathbf{Set}_{df}$ is the category of G-sets which are both discrete and finite. A family $U_\alpha \to X$ is covering if and only if $\bigsqcup U_\alpha \to X$ is surjective.

Main example: G is the profinite group $\{G(L/K)\}$ of Galois groups of the finite Galois extensions L/K of a field K.

9) Suppose that $\mathscr{C}$ is a small category. Say that $R \subset \hom(\ ,x)$ is covering if and only if $1_x \in R$. This is the **chaotic topology** on $\mathscr{C}$.

10) Suppose that $\mathscr{C}$ is a site and that $U \in \mathscr{C}$. The slice category $\mathscr{C}/U$ inherits a topology from $\mathscr{C}$: the set of maps $V_\alpha \to V \to U$ covers $V \to U$ if and only if the family $V_\alpha \to V$ covers V.

Definitions: Suppose that $\mathscr{C}$ is a Grothendieck site.

1) A **presheaf** (of sets) on $\mathscr{C}$ is a functor

$$\mathscr{C}^{op} \to \mathbf{Set}.$$

   If $\mathscr{A}$ is a category, an $\mathscr{A}$-valued presheaf on $\mathscr{C}$ is a functor $\mathscr{C}^{op} \to \mathscr{A}$.

The set-valued presheaves on $\mathscr{C}$ form a category (morphisms are natural transformation), written $\mathrm{Pre}(\mathscr{C})$.

One defines presheaves taking values in any category: I write $s\mathrm{Pre}(\mathscr{C})$ for presheaves on $\mathscr{C}$ taking values in simplicial sets — this is the category of **simplicial presheaves** on $\mathscr{C}$.

2) A **sheaf** (of sets) on $\mathscr{C}$ is a presheaf $F : \mathscr{C}^{op} \to$ **Set** such that the canonical map

$$F(U) \to \varprojlim_{V \to U \in R} F(V)$$

   is an isomorphism for each covering sieve $R \subset \hom(\ ,U)$.

Morphisms of sheaves are natural transformations: write $\mathrm{Shv}(\mathscr{C})$ for the corresponding category.

The **sheaf category** $\mathrm{Shv}(\mathscr{C})$ is a full subcategory of $\mathrm{Pre}(\mathscr{C})$.

One defines sheaves in any complete category, such as simplicial sets: $s\,\mathrm{Shv}(\mathscr{C})$ denotes the category of simplicial sheaves on the site $\mathscr{C}$.

Exercise: If the topology on $\mathscr{C}$ is defined by a pretopology (so that $\mathscr{C}$ has all pullbacks), show that F is a sheaf if and only if all pictures

$$F(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

arising from covering families $U_\alpha \to U$ are equalizers.

Lemma 5.1. *1) If $R \subset R' \subset \hom(\ ,U)$ and R is covering then R' is covering.*

2) If $R, R' \subset \hom(\ ,U)$ are covering then $R \cap R'$ is covering.

Proof. 1) $\phi^{-1}(R) = \phi^{-1}(R')$ for all $\phi \in R$.
2) $\phi^{-1}(R \cap R') = \phi^{-1}(R')$ for all $\phi \in R$. □

Suppose that $R \subset \hom(\ ,U)$ is a sieve, and F is a presheaf on $\mathscr{C}$. Write

$$F(U)_R = \varprojlim_{V \to U \in R} F(V)$$

I say that $F(U)_R$ is the set of R-**compatible families** in U.

If $S \subset R$ then there is an obvious restriction map

$$F(U)_R \to F(U)_S$$

Write

$$LF(U) = \varinjlim_{R} F(U)_R$$

where the colimit is indexed over the filtering diagram of all covering sieves $R \subset \hom(\ ,U)$. Then $x \mapsto LF(U)$ is a presheaf and there is a natural presheaf map

$$\eta : F \to LF$$

Say that a presheaf G is **separated** if (equivalently)

1) the map $\eta : G \to LG$ is monic in each section, ie. all functions $G(U) \to LG(U)$ are injective, or

2) Given $x, y \in G(U)$, if there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x) = \phi^*(y)$ for all $\phi \in R$, then $x = y$.

Lemma 5.2. *1) LF is separated, for all F.*

2) If G is separated then LG is a sheaf.

3) If $f : F \to G$ is a presheaf map and G is a sheaf, then f factors uniquely through a presheaf map $f_* : LF \to G$.

Proof. Exercise. □

Corollary 5.3. *1) The object L^2F is a sheaf for every presheaf F.*

2) The functor $F \mapsto L^2F$ is left adjoint to the inclusion $\mathrm{Shv}(\mathscr{C}) \subset \mathrm{Pre}(\mathscr{C})$. *The unit of the adjunction is the composite*

$$F \xrightarrow{\eta} LF \xrightarrow{\eta} L^2F \tag{5.1}$$

One often writes

$$\eta : F \to L^2F = \tilde{F}$$

for the composite (5.1).

$L^2F = \tilde{F}$ is the **associated sheaf** for F, and η is the canonical map.

6 Exactness properties

Lemma 6.1. *1) The associated sheaf functor preserves all finite limits.*

2) $\mathrm{Shv}(\mathscr{C})$ *is complete and co-complete. Limits are formed sectionwise.*

3) Every monic is an equalizer.

4) If $\theta : F \to G$ in $\mathrm{Shv}(\mathscr{C})$ *is both monic and epi, then θ is an isomorphism.*

Proof. 1) LF is defined by filtered colimits, and finite limits commute with filtered colimits.

2) If $X : I \to \mathrm{Shv}(\mathscr{C})$ is a diagram of sheaves, then the colimit in the sheaf category is $L^2(\varinjlim X)$, where $\varinjlim X$ is the presheaf colimit.

3) If $A \subset X$ is a subset, then there is an equalizer

$$A \longrightarrow X \underset{*}{\overset{p}{\rightrightarrows}} X/A$$

The same holds for subobjects $A \subset X$ of presheaves, and hence for subobjects of sheaves, since L^2 is exact.

4) The map θ appears in an equalizer

$$F \xrightarrow{\theta} G \underset{g}{\overset{f}{\rightrightarrows}} K$$

since θ is monic. θ is an epi, so $f = g$. But then $1_G : G \to G$ factors through θ, giving a section $\sigma : G \to F$. Finally, $\theta\sigma\theta = \theta$ and θ is monic, so $\sigma\theta = 1$. □

Definitions:

1) A presheaf map $f : F \to G$ is a **local epimorphism** if for each $x \in G(U)$ there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x) = f(y_\phi)$ for all $\phi : V \to U$ in R.

2) $f : F \to G$ is a **local monic** if given $x, y \in F(U)$ such that $f(x) = f(y)$, then there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x) = \phi^*(y)$ for all $\phi : V \to U$ in R.

3) A presheaf map $f : F \to G$ which is both a local epi and a local monic is a **local isomorphism**.

Lemma 6.2. *1) The natural map $\eta : F \to LF$ is a local monomorphism and a local epimorphism.*

2) Suppose that $f : F \to G$ is a presheaf morphism. Then f induces an isomorphism of associated sheaves if and only if f is both a local epi and a local monic.

Proof. For 2) observe that, given a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & F' \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle f} \\ & & F'' \end{array}$$

of presheaf morphisms, if any two of f, g and h are local isomorphisms, then so is the third.

A sheaf map $g : E \to E'$ is a monic (respectively epi) if and only if it is a local monic (respectively local epi). $\square$

A **Grothendieck topos** is a category $\mathscr{E}$ which is equivalent to a sheaf category $\mathrm{Shv}(\mathscr{C})$ on some Grothendieck site $\mathscr{C}$.

Grothendieck toposes are characterized by exactness properties:

Theorem 6.3 (Giraud)**.** *A category $\mathscr{E}$ having all finite limits is a Grothendieck topos if and only if it has the following properties:*

1) *$\mathscr{E}$ has all small coproducts; they are disjoint and stable under pullback*
2) *every epimorphism of $\mathscr{E}$ is a coequalizer*
3) *every equivalence relation $R \to E \times E$ in $\mathscr{E}$ is a kernel pair and has a quotient*
4) *every coequalizer $R \rightrightarrows E \to Q$ is stably exact*
5) *there is a (small) set of objects which generates $\mathscr{E}$.*

A sketch proof of Giraud's Theorem appears below, but the result is proved in many places — see, for example, [2], [3]. See also [1].

Here are the definitions of the terms appearing in the statement of the Theorem:

1) The coproduct $\bigsqcup_i A_i$ is **disjoint** if all diagrams

$$\begin{array}{ccc} \emptyset & \longrightarrow & A_j \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & \bigsqcup_i A_i \end{array}$$

are pullbacks for $i \neq j$. $\bigsqcup_i A_i$ is **stable under pullback** if all diagrams

$$\begin{array}{ccc} \bigsqcup_i B' \times_B A_i & \longrightarrow & \bigsqcup_i A_i \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

are pullbacks.

3) An **equivalence relation** is a monomorphism $m = (m_0, m_1) : R \to E \times E$ such that

a) the diagonal $\Delta : E \to E \times E$ factors through m (ie. $a \sim a$)

b) the composite $R \xrightarrow{m} E \times E \xrightarrow{\tau} E \times E$ factors through m (ie. $a \sim b \Rightarrow b \sim a$).

c) the map

$$(m_0 m_{0*}, m_1 m_{1*}) : R \times_E R \to R \times R$$

factors through m (this is transitivity) where the pullback is defined by

$$\begin{array}{ccc} R\times_E R & \xrightarrow{m_{1*}} & R \\ {\scriptstyle m_{0*}}\downarrow & & \downarrow{\scriptstyle m_0} \\ R & \xrightarrow[m_1]{} & E \end{array}$$

The **kernel pair** of a morphism $u : E \to D$ is a pullback

$$\begin{array}{ccc} R & \xrightarrow{m_1} & E \\ {\scriptstyle m_0}\downarrow & & \downarrow{\scriptstyle u} \\ E & \xrightarrow[u]{} & D \end{array}$$

(Exercise: every kernel pair is an equivalence relation).

A **quotient** for an equivalence relation $(m_0, m_1) : R \to E \times E$ is a coequalizer

$$R \underset{m_1}{\overset{m_0}{\rightrightarrows}} E \longrightarrow E/R$$

4) A coequalizer $R \rightrightarrows E \to Q$ is **stably exact** if the diagram

$$R \times_Q Q' \rightrightarrows E \times_Q Q' \to Q'$$

is a coequalizer for all morphisms $Q' \to Q$.

5) A **generating set** is a set $\{A_i\}$ which detects non-trivial monomorphisms: if a monomorphism $m : P \to Q$ induces bijections $\hom(A_i, P) \to \hom(A_i, Q)$ for all i, then m is an isomorphism.

Exercise: Show that any category $\mathrm{Shv}(C)$ on a site $\mathscr{C}$ satisfies the conditions of Giraud's theorem. The family $L^2\hom(\ ,U)$, $U \in \mathscr{C}$ is a set of generators.

Sketch proof of Giraud's Theorem. The key is to show that the category $\mathscr{E}$ is cocomplete — see [2].

If A is the set of generators for $\mathscr{E}$ prescribed by Giraud's theorem, let $\mathscr{C}$ be the full subcategory of $\mathscr{E}$ on the set of objects A. A subfunctor $R \subset \hom(\ ,x)$ on $\mathscr{C}$ is covering if the map

$$\bigsqcup_{y \to x \in R} y \to x$$

is an epimorphism of $\mathscr{E}$.

Every object $E \in \mathscr{E}$ represents a sheaf $\hom(\ ,E)$ on $\mathscr{C}$, and a sheaf F on $\mathscr{C}$ determines an object

$$\varinjlim_{\hom(\ ,y) \to F} y$$

of $\mathscr{E}$.

The adjunction

$$\hom(\varinjlim_{\hom(\ ,y) \to F} y, E) \cong \hom(F, \hom(\ ,E))$$

determines an adjoint equivalence between $\mathscr{E}$ and $\mathrm{Shv}(\mathscr{C})$. $\square$

The proof of Giraud's Theorem is arguably more important than the statement of the theorem itself. Here are some examples of the use of the basic ideas:

1) Suppose that G is a sheaf of groups, and let $G-\mathrm{Shv}(\mathscr{C})$ denote the category of all sheaves X admitting G-action, with equivariant maps between them.

$G-\mathrm{Shv}(\mathscr{C})$ is a Grothendieck topos, called the **classifying topos** for G, by Giraud's theorem. The objects $G\times \hom(\ ,U)$ form a generating set.

2) If $G=\{G_i\}$ is a profinite group with all transition maps $G_i\to G_j$ epi, then the category $G-\mathbf{Set}_d$ of discrete G-sets is a Grothendieck topos. The finite discrete G-sets form a generating set for this topos, and the site of finite discrete G-sets is "the" site prescribed by Giraud's theorem.

7 Geometric morphisms

Suppose that $\mathscr{C}$ and $\mathscr{D}$ are Grothendieck sites. A **geometric morphism** $f:\mathrm{Shv}(\mathscr{C})\to\mathrm{Shv}(\mathscr{D})$ consists of functors $f_*:\mathrm{Shv}(\mathscr{C})\to\mathrm{Shv}(\mathscr{D})$ and $f^*:\mathrm{Shv}(\mathscr{D})\to\mathrm{Shv}(\mathscr{C})$ such that

1) f^* is left adjoint to f_*, and

2) f^* preserves finite limits.

The left adjoint f^* is called the **inverse image** functor, while f_* is called the **direct image**.

The inverse image f^* is left and right exact in the sense that it preserves all finite limits and colimits.

The direct image f_* is usually not left exact (does not preserve finite colimits), and hence has higher derived functors.

Examples

1) Suppose $f : X \to Y$ is a continuous map of topological spaces. Pullback along f induces a functor $\mathrm{op}|_Y \to \mathrm{op}|_X$: $U \subset Y \mapsto f^{-1}(U)$.

Open covers pull back to open covers, so if F is a sheaf on X then composition with the pullback gives a sheaf f_*F on Y with $f_*F(U) = F(f^{-1}(U))$.

The resulting functor $f_* : \mathrm{Shv}(\mathrm{op}|_X) \to \mathrm{Shv}(\mathrm{op}|_Y)$ is the direct image

The **left Kan extension** $f^p : \mathrm{Pre}(\mathrm{op}|_Y) \to \mathrm{Pre}(\mathrm{op}|_X)$ is defined by

$$f^p G(V) = \varinjlim G(U)$$

where the colimit is indexed over all diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

The category $\mathrm{op}|_Y$ has all products (ie. intersections), so the colimit is filtered. The functor $G \mapsto f^p G$ therefore commutes with finite limits. The inverse image functor

$$f^* : \mathrm{Shv}(\mathrm{op}|_Y) \to \mathrm{Shv}(\mathrm{op}|_X)$$

is defined by $f^*(G) = L^2 f^p(G)$.

The resulting pair of functors f_*, f^* forms a geometric morphism $f : \mathrm{Shv}(\mathrm{op}|_X) \to \mathrm{Shv}(\mathrm{op}|_Y)$.

2) Suppose $f : X \to Y$ is a morphism of schemes.

Etale maps (resp. covers) are stable under pullback, and so there is a functor $\mathrm{et}|_Y \to \mathrm{et}|_X$ defined by pullback, and if F is a sheaf on $\mathrm{et}|_X$ then there is a sheaf f_*F on $\mathrm{et}|_Y$ defined by

$$f_*F(V \to Y) = F(X \times_Y V \to X).$$

The restriction functor $f_* : \mathrm{Pre}(\mathrm{et}|_X) \to \mathrm{Pre}(\mathrm{et}|_Y)$ has a left adjoint f^p defined by

$$f^p G(U \to X) = \varinjlim G(V \to Y)$$

where the colimit is indexed over all diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

where both vertical maps are étale. The colimit is filtered, because étale maps are stable under pullback and composition. The inverse image functor

$$f^* : \mathrm{Shv}(\mathrm{et}|_Y) \to \mathrm{Shv}(et|_X)$$

is defined by $f^*F = L^2 f^p F$, and so f induces a geometric morphism $f : \mathrm{Shv}(\mathrm{et}|_X) \to \mathrm{Shv}(\mathrm{et}|_Y)$.

A morphism of schemes $f : X \to Y$ induces a geometric morphism $f : \mathrm{Shv}(?|_X) \to \mathrm{Shv}(?|_Y)$ and/or $f : (Sch|_X)_? \to (Sch|_Y)_?$ for all of the geometric topologies (eg. Zariski, flat, Nisnevich, qfh, ...), by similar arguments.

3) A **point** of $\mathrm{Shv}(\mathscr{C})$ is a geometric morphism $\mathbf{Set} \to \mathrm{Shv}(\mathscr{C})$.

Every point $x \in X$ of a topological space X determines a continuous map $\{x\} \subset X$ and hence a geometric morphism

$$\mathbf{Set} \cong \mathrm{Shv}(\mathrm{op}|_{\{x\}}) \xrightarrow{x} \mathrm{Shv}(\mathrm{op}|_X)$$

The set

$$x^*F = \varinjlim_{x \in U} F(U)$$

is the **stalk** of F at x.

The object x_*Z associated to a set Z is called a **skyscraper sheaf**.

4) Suppose that k is a field. A scheme map $x : \text{Sp}(k) \to X$ induces a geometric morphism

$$\text{Shv}(et|_k) \to \text{Shv}(et|_X)$$

If k happens to be separably closed, then there is an equivalence $\text{Shv}(et|_k) \simeq \textbf{Set}$ and the resulting geometric morphism $x : \textbf{Set} \to \text{Shv}(\text{et}|_X)$ is called a **geometric point** of X. The inverse image functor

$$F \mapsto f^*F = \varinjlim_{\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ \text{Sp}(k) & \xrightarrow[x]{} & X \end{array}} F(U)$$

is the stalk of F at x.

5) Suppose that S and T are topologies on a site $\mathscr{C}$ so that $S \subset T$. In other words, T has more covers than S and hence refines S. Then every sheaf for T is a sheaf for S. Write

$$\pi_* : \text{Shv}(\mathscr{C}, T) \subset \text{Shv}(\mathscr{C}, S)$$

for the corresponding inclusion.

The associated sheaf functor for the topology T gives a left adjoint π^* for the inclusion functor π_*, and π^* preserves finite limits.

Example: There is a geometric morphism

$$\mathrm{Shv}(\mathscr{C}) \to \mathrm{Pre}(\mathscr{C})$$

determined by the inclusion of the sheaf category in the presheaf category and the associated sheaf functor.

8 Points and Boolean localization

A Grothendieck topos $\mathrm{Shv}(\mathscr{C})$ has **enough points** if there is a set of geometric morphisms $x_i : \mathbf{Set} \to \mathrm{Shv}(\mathscr{C})$ such that the induced morphism

$$\mathrm{Shv}(\mathscr{C}) \xrightarrow{(x_i^*)} \prod_i \mathbf{Set}$$

is faithful.

Lemma 8.1. *Suppose that* $f : \mathrm{Shv}(\mathscr{D}) \to \mathrm{Shv}(\mathscr{C})$ *is a geometric morphism. Then the following are equivalent:*

a) $f^* : \mathrm{Shv}(\mathscr{C}) \to \mathrm{Shv}(\mathscr{D})$ *is faithful.*

b) f^* *reflects isomorphisms*

c) f^* *reflects epimorphisms*

d) f^* *reflects monomorphisms*

Proof. Suppose that f^* is faithful, ie. that $f^*(g_1) = f^*(g_2)$ implies that $g_1 = g_2$.

Suppose that $m : F \to G$ is a morphism of $\mathrm{Shv}(\mathscr{C})$ such that $f^*(m)$ is monic. If $m \cdot f_1 = m \cdot f_2$ then $f^*(f_1) = f^*(f_2)$ so $f_1 = f_2$. The map m is therefore monic.

Similarly, f^* reflects epimorphisms and hence reflects isomorphisms.

Suppose that f^* reflects epis and suppose given $g_1, g_2 : F \to G$ such that $f^*(g_1) = f^*(g_2)$.

$g_1 = g_2$ if and only if their equalizer $e : E \to F$ is an isomorphism. But f^* preserves equalizers and reflects isomorphisms, so e is an epi and $g_1 = g_2$.

The other arguments are similar. □

Here are some basic definitions:

1) A **lattice** L is a partially ordered set which has coproducts $x \vee y$ and products $x \wedge y$.

2) A lattice L has 0 and 1 if it has an initial and terminal object, respectively.

3) A lattice L is said to be **distributive** if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z.

4) Suppose that L is a lattise with 0 and 1 and that $x \in L$. A **complement** for x is an element a such that $x \vee a = 1$ and $x \wedge a = 0$.

If L is also distibutive the complement, if it exists, is unique: if b is another complement for x, then

$$\begin{aligned} b = b \wedge 1 = b \wedge (x \vee a) &= (b \wedge x) \vee (b \wedge a) \\ &= (x \wedge a) \vee (b \wedge a) = (x \vee b) \wedge a = a \end{aligned}$$

One usually writes $\neg x$ for the complement of x.

5) A **Boolean algebra** $\mathscr{B}$ is a distributive lattice with 0 and 1 in which every element has a complement.

6) A lattice L is said to be **complete** if it has all small limits and colimits (aka. all small meets and joins).

7) A **frame** P is a lattice that has all small joins and satisfies an infinite distributive law

$$U \wedge (\bigvee_i V_i) = \bigvee_i (U \wedge V_i)$$

Examples:

1) The poset $\mathscr{O}(T)$ of open subsets of a topological space T is a frame.
2) The power set $\mathscr{P}(I)$ of a set I is a complete Boolean algebra.
3) Every complete Boolean algebra $\mathscr{B}$ is a frame. In effect, every join is a filtered colimit of finite joins.

Every frame A has a canonical Grothendieck topology: a family $y_i \leq x$ is covering if $\bigvee_i y_i = x$. Write $\mathrm{Shv}(A)$ for the corresponding sheaf category.

Every complete Boolean algebra $\mathscr{B}$ is a frame, and has an associated sheaf category $\mathrm{Shv}(\mathscr{B})$.

Example: Suppose that I is a set. Then there is an equivalence

$$\text{Shv}(\mathscr{P}(I)) \simeq \prod_{i \in I} \mathbf{Set}$$

Any set I of points $x_i : \mathbf{Set} \to \text{Shv}(\mathscr{C})$ assembles to give a geometric morphism

$$x : \text{Shv}(\mathscr{P}(I)) \to \text{Shv}(\mathscr{C}).$$

Here

$$x(F_i) = \prod_{i \in I} x_{i*}(F_i).$$

Lemma 8.2. *Suppose that F is a sheaf of sets on a complete Boolean algebra $\mathscr{B}$. Then the poset $Sub(F)$ of subobjects of F is a complete Boolean algebra.*

Proof. $Sub(F)$ is a frame, by an argument on the presheaf level. It remains to show that every object $G \in Sub(F)$ is complemented. The obvious candidate for $\neg G$ is

$$\neg G = \bigvee_{H \wedge G = \emptyset} H$$

and we need to show that $G \bigvee \neg G = F$.

Every $K \leq \hom(\ ,A)$ is representable: in effect,

$$K = \varinjlim_{\hom(\ ,B) \to K} \hom(\ ,B) = \hom(\ ,C)$$

where

$$C = \bigvee_{\hom(\ ,B)\to K} B \in \mathscr{B}.$$

It follows that $Sub(\hom(\ ,A)) \cong Sub(A)$ is a complete Boolean algebra.

Consider all diagrams

$$\begin{array}{ccc} \phi^{-1}(G) & \longrightarrow & G \\ \downarrow & & \downarrow \\ \hom(\ ,A) & \xrightarrow[\phi]{} & F \end{array}$$

There is an induced pullback

$$\begin{array}{ccc} \phi^{-1}(G) \vee \neg\phi^{-1}(G) & \longrightarrow & G \vee \neg G \\ \cong\downarrow & & \downarrow \\ \hom(\ ,A) & \xrightarrow[\phi]{} & F \end{array}$$

F is a union of its representables (all ϕ are monic since all $\hom(\ ,A)$ are subobjects of the terminal sheaf), so $G \vee \neg G = F$. $\square$

Lemma 8.3. *Suppose that $\mathscr{B}$ is a complete Boolean algebra. Then every epimorphism $\pi : F \to G$ in* $\mathrm{Shv}(\mathscr{B})$ *has a section.*

Remark 8.4. Lemma 8.3 asserts that the sheaf category on a complete Boolean algebra satisfies the **Axiom of Choice**.

Proof of Lemma 8.3. Consider the family of lifts

$$\begin{array}{ccc} & & F \\ & \nearrow & \downarrow \pi \\ N & \xrightarrow[\leq]{} & G \end{array}$$

This family is non-empty, because every $x \in G(1)$ restricts along some covering $B \leq 1$ to a family of elements x_B which lift to $F(B)$.

All maps $\hom(\ ,B) \to G$ are monic, since all maps $\hom(\ ,B) \to \hom(\ ,1) = *$ are monic. Thus, all such morphisms represent objects of $Sub(G)$, which is a complete Boolean algebra by Lemma 8.2.

Zorn's Lemma implies that the family of lifts has maximal elements.

Suppose that N is maximal and that $\neg N \neq \emptyset$. Then there is an $x \in \neg N(C)$ for some C, and there is a cover $B' \leq C$ such that $x_{B'} \in N(B')$ lifts to $F(B')$ for all B'. Then $N \wedge \hom(\ ,B') = \emptyset$ so the lift extends to a lift on $N \vee \hom(\ ,B')$, contradicting the maximality of N. □

A **Boolean localization** for $\mathrm{Shv}(\mathscr{C})$ is a geometric morphism $p : \mathrm{Shv}(\mathscr{B}) \to \mathrm{Shv}(\mathscr{C})$ such that $\mathscr{B}$ is a complete Boolean algebra and p^* is faithful.

Theorem 8.5 (Barr). *Boolean localizations exist for every Grothendieck topos* $\mathrm{Shv}(\mathscr{C})$.

Theorem 8.5 is one of the big results of topos theory, and is proved in multiple places — see [2], for example. There is a relatively simple description of the proof in [1].

In general, a Grothendieck topos $\mathrm{Shv}(\mathscr{C})$ does not have enough points (eg. sheaves on the flat site for a scheme), but Theorem 8.5 asserts that every Grothendieck topos has a "fat point" given by a Boolean localization.

References

[1] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.

[2] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[3] Horst Schubert. *Categories*. Springer-Verlag, New York, 1972. Translated from the German by Eva Gray.