22 Localization for simplicial presheaves

Suppose that $\mathscr{C}$ is a small Grothendieck site, and S is a set of cofibrations $A \to B$ in $s\mathrm{Pre}(\mathscr{C})$.

Assume throughout this section that I is a simplicial presheaf on $\mathscr{C}$ with disjoint global sections $0, 1 : * \to I$.

The object I will be called an **interval**, whether it looks like one or not.

S will be a fixed set of cofibrations of $s\mathrm{Pre}(\mathscr{C})$.

Examples include the following:

1) the simplicial set Δ^1 with the two vertices $0, 1 : * \to \Delta^1$,
2) $B\pi(\Delta^1)$ with the two vertices $0, 1 : * \to \pi(\Delta^1)$. $\pi(\Delta^1)$ is the fundamental groupoid of Δ^1,
3) the affine line $\mathbb{A}^1$ over a scheme T with the rational points $0, 1 : T \to \mathbb{A}^1$.

The idea of localization theory is to construct, in a minimal way, a homotopy theory on $s\mathrm{Pre}(\mathscr{C})$ for which the cofibrations are the monomorphisms, all maps in S become weak equivalences, and the interval I describes homotopies.

Write

$$\square^n = I^{\times n}.$$

There are face inclusions

$$d^{i,\varepsilon} : \square^{n-1} \to \square^n,\ 1 \leq i \leq n,\ \varepsilon = 0,1,$$

with

$$d^{i,\varepsilon}(x_1,\ldots,x_{n-1}) = (x_1,\ldots,x_{i-1},\varepsilon,x_i,\ldots,x_{n-1}).$$

Then there are subobjects $\partial\square^n$ and $\sqcap^n_{i,\varepsilon}$ of $\square^n$ which are defined, respectively, by

$$\partial\square^n = \cup_{i,\varepsilon}\, d^{i,\varepsilon}(\square^{n-1}),$$

and

$$\sqcap^n_{i,\varepsilon} = \cup_{(j,\gamma)\neq(i,\varepsilon)}\, d^{j,\gamma}(\square^{n-1}).$$

A **naive homotopy** between maps $f,g : X \to Y$ is a commutative diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle 0}\downarrow & \searrow^{f} & \\ X \times I & \xrightarrow{h} & Y \\ {\scriptstyle 1}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

Naive homotopies generate an equivalence relation: write

$$\pi(X,Y) = \pi_I(X,Y)$$

for the naive homotopy classes of maps $X \to Y$.

The class of **anodyne cofibrations** (or anodyne extensions) is the saturation of the set of inclusions $\Lambda(S)$ specified by

$$(C \times \square^n) \cup (D \times \sqcap^n_{(i,\varepsilon)}) \subset D \times \square^n \qquad (22.1)$$

where $C \to D$ is a member of a set of generating cofibrations for $s\,\mathrm{Pre}(\mathscr{C})$, and

$$(A \times \square^n) \cup (B \times \partial \square^n) \subset B \times \square^n \qquad (22.2)$$

with $A \to B$ in the set S.

An **injective morphism** is a simplicial presheaf map $p : X \to Y$ which has the right lifting property with respect to all anodyne extensions.

A simplicial presheaf X is **injective** if the map $X \to *$ is an injective morphism.

A **weak equivalence** is a map $f : X \to Y$ which induces a bijection $\pi(Y,Z) \to \pi(X,Z)$ for all injective Z.

A **cofibration** is a monomorphism.

A **fibration** is a map which has the right lifting property with respect to all trivial cofibrations.

Exercises:

1) Show that naive homotopy of maps $X \to Z$ is an equivalence relation if Z is injective.

2) Show that a map $f : Z \to W$ of injective objects is a weak equivalence if and only if it is a naive homotopy equivalence.

This means that there is a map $g : W \to Z$ and naive homotopies $f \cdot g \simeq 1_W$ and $g \cdot f \simeq 1_Z$.

Lemma 22.1. *1) Suppose that $C \to D$ is an anodyne cofibration. Then the induced map*

$$(C \times \square^1) \cup (D \times \partial \square^1) \subset D \times \square^1 \tag{22.3}$$

is anodyne.

2) All anodyne cofibrations are weak equivalences.

Proof. Show that that if $C \to D$ is in $\Lambda(S)$, then the induced map (22.3) is in $\Lambda(S)$.

Then the proof of statement 1) is finished with a colimit argument.

For 2), suppose that $i : C \to D$ is an anodyne cofibration and that Z is an injective object.

Then the lifting exists in any diagram

$$\begin{array}{ccc} C & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & \nearrow & \\ D & & \end{array}$$

so that the map

$$i^* : \pi(D,Z) \to \pi(C,Z)$$

is surjective. If $f,g : D \to Z$ are morphisms such that there is a homotopy $h : C \times I \to Z$ between fi and gi, then the lifting exists in the diagram

$$\begin{array}{ccc} (C \times \square^1) \cup (D \times \partial\square^1) & \xrightarrow{(h,(f,g))} & Z \\ \downarrow & \nearrow_{H} & \\ D \times \square^1 & & \end{array}$$

(by part 1)) and the map H is a homotopy between f and g. It follows that the function

$$i^* : \pi(D,Z) \to \pi(C,Z)$$

is injective. □

We shall sketch the proof of the following:

Theorem 22.2 (Cisinski)**.** *With the definitions given above, the simplicial presheaf category $s\mathrm{Pre}(\mathscr{C})$ has the structure of a left proper cubical model category.*

The cubical model structure involves the cubical set (the **cubical function complex**)

$$\mathbf{hom}(X,Y)$$

whose n-cells are the maps $X \times \square^n \to Y$.

This construction satisfies a cubical version of Quillen's simplicial model axiom **SM7** — the proof is built into the proof of Theorem 22.2.

There is a properness assertion as well:

Theorem 22.3. *Suppose that all cofibrations in the set S pull back to weak equivalences along all fibrations $p : X \to Y$ with Y fibrant.*

Then the model structure of Theorem 22.2 on $s\mathrm{Pre}(\mathscr{C})$ is proper.

The condition in the statement of Theorem 22.3 means that, in every diagram

$$\begin{array}{ccccc} A \times_Y X & \xrightarrow{i_*} & B \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ A & \xrightarrow[i]{} & B & \longrightarrow & Y \end{array}$$

with p a fibration and Y fibrant, if i is a member of S then i_* is a weak equivalence.

Theorems 22.2 and 22.3 are special cases of more general results, which can be found in [5] and (better) [6].

Theorem 22.2 was originally proved by Cisinski [1]. He did not express the result as it appears here, but the main ideas of the proof are due to him.

1) Cardinality tricks

Suppose that T is some set of cofibrations, and choose a regular cardinal α such that $\alpha > |T|$ and that $\alpha > |D|$ for all $C \to D$ in T. Suppose that $\alpha > |\mathrm{Mor}(\mathscr{C})|$.

Suppose that $\lambda > 2^{\alpha}$ is regular.

Every $f : X \to Y$ has a functorial system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & E_s(f) \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle f_s} \\ & & Y \end{array}$$

for $s < \lambda$ defined by the lifting property for maps in T, and which form the stages of a transfinite small object argument.

Specifically, given the factorization $f = f_s i_s$ form the pushout diagram

$$\begin{array}{ccc} \bigsqcup_{\mathscr{D}} C & \longrightarrow & E_s(f) \\ \downarrow & & \downarrow \\ \bigsqcup_{\mathscr{D}} D & \longrightarrow & E_{s+1}(f) \end{array}$$

where $\mathscr{D}$ runs through all diagrams

$$\begin{array}{ccc} C & \longrightarrow & E_s(f) \\ {\scriptstyle i} \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

with i in T.

Then $f_{s+1} : E_{s+1}(f) \to Y$ is the obvious induced map.

Set $E_t(f) = \varinjlim_{s<t} E_s(f)$ at limit ordinals $t < \lambda$.

Then there is a functorial factorization

$$\begin{array}{ccc} X & \xrightarrow{i_\lambda} & E_\lambda(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_\lambda} \\ & & Y \end{array}$$

with $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$.

The map f_λ has the right lifting property with respect to all $C \to D$ in T, and i_λ is in the saturation of T.

Write $\mathscr{L}(X) = E_\lambda(X \to *)$.

Lemma 22.4. *1) Suppose that $t \mapsto X_t$ is a diagram of simplicial presheaves, indexed by $\omega > 2^\alpha$.*

Then the map

$$\varinjlim_{t<\omega} \mathscr{L}(X_t) \to \mathscr{L}(\varinjlim_{t<\omega} X_t)$$

is an isomorphism.

2) The functor $X \mapsto \mathscr{L}(X)$ preserves cofibrations.

3) Suppose that γ is a cardinal with $\gamma > \alpha$, and let $\mathscr{F}_\gamma(X) =$ the subobjects of X having cardinality less than γ.

Then the map

$$\varinjlim_{Y \in \mathscr{F}_\gamma(X)} \mathscr{L}(Y) \to \mathscr{L}(X)$$

is an isomorphism.

4) If $|X| \leq 2^\mu$ where $\mu \geq \lambda$ then $|\mathscr{L}(X)| \leq 2^\mu$.

5) Suppose that U,V are subobjects of X.

Then the natural map

$$\mathscr{L}(U \cap V) \to \mathscr{L}(U) \cap \mathscr{L}(V)$$

is an isomorphism.

Proof. It suffices to prove all statements with $\mathscr{L}(X)$ replaced by $E_1(X)$.

There is a pushout diagram

$$\begin{array}{ccc} \bigsqcup_T (C \times \hom(C,X)) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bigsqcup_T (D \times \hom(C,X)) & \longrightarrow & E_1 X \end{array}$$

Then, in sections,

$$E_1X(a) = \bigsqcup_T ((D(a) - C(a)) \times \hom(C,X)) \sqcup X(a),$$

so 5) follows.

The remaining statements are exercises. □

Corollary 22.5. *Every simplicial presheaf map $f : X \to Y$ has a functorial factorization*

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle p} \\ & & Y \end{array}$$

where j is anodyne and p is injective.

Suppose that α is a regular cardinal such that $\alpha > |\Lambda(S)|$ and that $\alpha > |D|$ for all $C \to D$ in $\Lambda(S)$. Suppose that $\alpha > |\operatorname{Mor}(\mathscr{C})|$. Suppose that $\lambda > 2^\alpha$.

Here is the bounded cofibration condition:

Lemma 22.6. *Suppose given a diagram*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \rightarrow & Y \end{array}$$

of cofibrations such that i is a weak equivalence and $|A| \leq 2^\lambda$.

Then there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| \leq 2^\lambda$ and $B \cap X \to B$ is an equivalence.

Proof. The proof is due to Cisinski. The innovation is that it uses nothing but naive homotopy.

The map $i_* : \mathscr{L}X \to \mathscr{L}Y$ is a cofibration (by Lemma 22.4) and is a naive homotopy equivalence of injective objects.

There is a map $\sigma : \mathscr{L}Y \to \mathscr{L}X$ such that $\sigma \cdot i_* \simeq 1$ via a naive homotopy $h : \mathscr{L}X \times \square^1 \to \mathscr{L}X$.

Form the diagram

$$\begin{array}{ccc} (\mathscr{L}Y \times \square^0) \cup (\mathscr{L}X \times \square^1) & \xrightarrow{(\sigma,h)} & \mathscr{L}X \\ \downarrow & \overset{H}{\nearrow} & \\ \mathscr{L}Y \times \square^1 & & \end{array}$$

The other end of the homotopy H gives a map σ' such that $\sigma' \cdot i_* = 1$, and $i_*\sigma' \simeq i_*\sigma \simeq 1$.

We can therefore assume that $\sigma \cdot i_* = 1$.

Suppose that $A_s \subset Y$ and $|A_s| \leq 2^\lambda$.

Then $|\mathscr{L}A_s \times \square^1| \leq 2^\lambda$.

Also, there is a subobject A_{s+1} such that $A_s \subset A_{s+1}$ with $|A_{s+1}| \leq 2^\lambda$, and there is a diagram

$$\begin{array}{ccc} \mathscr{L}A_s \times \square^1 & \rightarrow & \mathscr{L}A_{s+1} \\ \downarrow & & \downarrow \\ \mathscr{L}Y \times \square^1 & \xrightarrow[K]{} & \mathscr{L}Y \end{array}$$

where K is the homotopy $i_*\sigma \simeq 1$.

This is the successor ordinal step in the construction of a system $s \mapsto A_s$ with $s < \lambda$ (recall that $\lambda > 2^\alpha$) and $A = A_0$.

Let $B = \varinjlim_s A_s$.

Then, by construction, $|B| \leq 2^\lambda$, and the restriction of the homotopy K to $\mathscr{L}B \times \square^1$ factors through the inclusion $j_* : \mathscr{L}B \to \mathscr{L}Y$.

There is a pullback

$$\begin{array}{ccc} \mathscr{L}(B \cap X) & \xrightarrow{\tilde{j}} & \mathscr{L}X \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle i_*} \\ \mathscr{L}B & \xrightarrow[j_*]{} & \mathscr{L}Y \end{array}$$

and $i_*\sigma(\mathscr{L}B) \subset \mathscr{L}B$.

It follows that there is a map

$$\sigma' : \mathscr{L}B \to \mathscr{L}(B \cap X)$$

such that $\sigma' \cdot \tilde{i} = 1$.

The map K factors through a homotopy $\tilde{i}\sigma' \simeq 1$. $\square$

2) Trivial cofibrations are preserved by pushout

The class of anodyne extensions is closed under pushout, by definition.

Lemma 22.7. *Suppose given a diagram*

$$\begin{array}{ccc} C & \xrightarrow{f,g} & E \\ {\scriptstyle i}\downarrow & & \\ D & & \end{array}$$

where i is a cofibration, and suppose that there is a naive homotopy $h : C \times \square^1 \to E$ from f to g.

Then $g_* : D \to D \cup_g E$ is a weak equivalence if and only if $f_* : D \to D \cup_f E$ is a weak equivalence.

Proof. There are pushout diagrams

$$\begin{array}{ccccc} C & \xrightarrow{d_0} & C \times \square^1 & \xrightarrow{h} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ D & \xrightarrow[d_{0*}]{} & D \cup_C (C \times \square^1) & \xrightarrow[h']{} & D \cup_f E \\ & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_*} \\ & & D \times \square^1 & \xrightarrow[h_*]{} & (D \times \square^1) \cup_h E \end{array}$$

where the top composite is f.

The maps d_{0*}, j and j_* are anodyne cofibrations.

Thus $f_* = h' \cdot d_{0*}$ is a weak equivalence if and only if h' is a weak equivalence, and h' is a weak equivalence if and only if h_* is a weak equivalence.

Thus, f_* is a weak equivalence if and only if h_* is a weak equivalence.

Similarly, g_* is a weak equivalence if and only if h_* is a weak equivalence. □

Lemma 22.8. *Suppose that $i : C \to D$ is a trivial cofibration.*

Then the cofibration

$$(C \times \square^1) \cup (D \times \partial\square^1) \to D \times \square^1$$

is a weak equivalence.

Proof. The diagram

$$\begin{array}{ccccc} C \times \partial\square^1 & \to & D \times \partial\square^1 & \to & \mathscr{L}D \times \partial\square^1 \\ \downarrow & & \downarrow & & \downarrow \\ C \times \square^1 & \longrightarrow & D \times \square^1 & \longrightarrow & \mathscr{L}D \times \square^1 \end{array}$$

induces a diagram

$$\begin{array}{ccc} (C \times \square^1) \cup (D \times \partial\square^1) & \to & (C \times \square^1) \cup (\mathscr{L}D \times \partial\square^1) \\ \downarrow & & \downarrow \\ D \times \square^1 & \longrightarrow & \mathscr{L}D \times \square^1 \end{array}$$

in which the horizontal maps are anodyne extensions, and hence weak equivalences.

There is a factorization

$$\begin{array}{ccc} C & \xrightarrow{i'} & D' \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle p} \\ & & D \end{array}$$

where i' is anodyne and p is both injective and a weak equivalence.

In the induced diagram

$$\begin{array}{ccc} (C \times \square^1) \cup (\mathscr{L}D' \times \partial\square^1) & \to & (C \times \square^1) \cup (\mathscr{L}D \times \partial\square^1) \\ \downarrow & & \downarrow \\ \mathscr{L}D' \times \square^1 & \longrightarrow & \mathscr{L}D \times \square^1 \end{array}$$

the top horizontal map is induced by the homotopy equivalence

$$\mathscr{L}D' \times \partial\square^1 \to \mathscr{L}D \times \partial\square^1,$$

and is therefore an equivalence by Lemma 22.7.

The bottom horizontal map is also a homotopy equivalence.

The left hand vertical map is an equivalence by comparison with the map

$$(C \times \square^1) \cup (D' \times \partial\square^1) \to D' \times \square^1$$

which is an anodyne extension. $\square$

Lemma 22.9. *The class of trivial cofibrations is closed under pushout.*

Proof. If $j : C \to D$ is a cofibration and a weak equivalence, then every map $\alpha : C \to Z$ with Z injective extends to a map $D \to Z$.

In effect, there is a homotopy $h : C \times \square^1 \to Z$ from α to a map $\beta \cdot j$ for some map $\beta : D \to Z$, and then the homotopy extends:

$$\begin{array}{ccc} (C \times \square^1) \cup (D \times \{1\}) & \xrightarrow{(h,\beta)} & Z \\ \downarrow & \nearrow_{H} & \\ D \times \square^1 & & \end{array}$$

Note that the vertical map is an anodyne extension.

Now suppose given a pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ D & \longrightarrow & D' \end{array}$$

Then the diagram

$$\begin{array}{ccc} (C \times \square^1) \cup (D \times \partial\square^1) & \to & (C' \times \square^1) \cup (D' \times \partial\square^1) \\ \downarrow & & \downarrow \\ D \times \square^1 & \longrightarrow & D' \times \square^1 \end{array}$$

is a pushout.

The left vertical map is a trivial cofibration by Lemma 22.8, and therefore has the left lifting property with respect to the map $Z \to *$.

Thus, if two maps $f, g : D' \to Z$ restrict to homotopic maps on C', then $f \simeq g$. $\square$

3) Many injective maps are fibrations

Lemma 22.10. *Suppose that the map $p : X \to Y$ is injective and that Y is injective.*

Then p is a fibration.

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array} \tag{22.4}$$

where i is a trivial cofibration.

There is a map $\theta : B \to X$ such that $\theta \cdot i = \alpha$ since X is injective.

The constant homotopy $A \times \square^1 \xrightarrow{pr} A \xrightarrow{\alpha} X$ extends to a homotopy $h : B \times \square^1 \to Y$ as in the diagram

$$\begin{array}{ccc} (A \times \square^1) \cup (B \times \partial\square^1) & \xrightarrow{(p\alpha pr_A, (\beta, p\theta))} & Y \\ \downarrow & \nearrow_{h} & \\ B \times \square^1 & & \end{array}$$

since the vertical map is a trivial cofibration (Lemma 22.8) and Y is injective.

It follows that there is a homotopy

$$\begin{array}{ccc} A \times \square^1 & \xrightarrow{\alpha pr_A} & X \\ {\scriptstyle i\times i}\downarrow & & \downarrow{\scriptstyle p} \\ B \times \square^1 & \xrightarrow[h]{} & Y \end{array}$$

from the original diagram to a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[p\theta]{} & Y \end{array}$$

Find the indicated lifting in the diagram

$$\begin{array}{ccc} (A \times \square^1) \cup B & \xrightarrow{(\alpha pr_A, \theta)} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B \times \square^1 & \xrightarrow[h]{} & Y \end{array}$$

to show that the required lifting exists for the original diagram (22.4). $\square$

Corollary 22.11. *Every injective object is fibrant.*

4) Final approach

Lemma 22.12 (CM4). *Suppose that $p : X \to Y$ is a fibration and a weak equivalence.*

Then p has the right lifting property with respect to all cofibrations.

Proof. Suppose first that Y is injective.

Then p is a naive homotopy equivalence, and has a section $\sigma : Y \to X$ (exercise).

The map σ is a trivial cofibration so the lift exists in the diagram

$$\begin{array}{ccc} (Y \times \square^1) \cup (X \times \partial\square^1) & \xrightarrow{(\sigma\cdot pr,(1_X,\sigma\cdot p))} & X \\ \downarrow & \overset{H}{\nearrow} & \downarrow p \\ X \times \square^1 & \xrightarrow[p\times 1]{} Y \times \square^1 \xrightarrow[pr]{} & Y \end{array}$$

since the left vertical map is a weak equivalence by Lemma 22.8.

It follows that the identity diagram on $p : X \to Y$ is naively homotopic to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma\cdot p} & X \\ {\scriptstyle p}\downarrow & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[1]{} & Y \end{array}$$

Thus, any diagram

$$\begin{array}{ccc} A & \rightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ B & \rightarrow & Y \end{array}$$

is naively homotopic to a diagram which admits a lifting.

It follows that p has the right lifting property with respect to all cofibrations.

If Y is not injective, form the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[j_Y]{} & \mathscr{L}(Y) \end{array}$$

where j is an anodyne cofibration, q is injective, and j is an injective model for X.

Then q is a fibration by Lemma 22.10 and is a weak equivalence, so that q has the right lifting property with respect to all cofibrations, by the previous paragraphs.

Factorize the map $X \to Y \times_{\mathscr{L}(Y)} Z$ as

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & Y \times_{\mathscr{L}(Y)} Z \end{array}$$

where π has the right lifting property with respect to all cofibrations and i is a cofibration.

Write q_* for the induced map $Y \times_{\mathscr{L}(Y)} Z \to Y$.

Then the composite $q_*\pi$ has the right lifting property with respect to all cofibrations and is therefore a homotopy equivalence.

The cofibration i is also a weak equivalence, so the lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ i\downarrow & \nearrow & \downarrow p \\ Z & \xrightarrow[q_*\pi]{} & Y \end{array}$$

and p is a retract of a map which has the right lifting property with respect to all cofibrations. □

Corollary 22.13. *A map $p : X \to Y$ is a fibration and a weak equivalence if and only if it has the right lifting property with respect to all cofibrations.*

Proof. If p has the right lifting property with respect to all cofibrations, then p is a homotopy equivalence (exercise).

The converse is Lemma 22.12. □

Proof of Theorem 22.2. The cofibration/trivial fibration factorization statement of **CM5** and **CM4** are consequences of Corollary 22.13: every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{i}\searrow \quad \nearrow_{p} & \\ & W & \end{array}$$

where i is a cofibration and p has the right lifting property with respect to all cofibrations.

The trivial cofibration/fibration factorization statement follows from the bounded cofibration condition: every $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{j}\searrow \quad \nearrow_{q} & \\ & Z & \end{array}$$

where j is a cofibration and a weak equivalence and q is a fibration.

In order to conclude that j is a weak equivalence, we need to know that trivial cofibrations are closed under pushout, but this is Lemma 22.9.

All simplicial presheaves are cofibrant for the present model structure.

Left properness follows from general nonsense about categories of cofibrant objects — see [2, II.8.5].

□

Examples:

1) Homotopy theory of simplicial presheaves

Suppose that S is a generating set of trivial cofibrations $A \to B$ for the injective model structure on $s\mathrm{Pre}(\mathscr{C})$, and that $I = \Delta^1$ is the standard interval.

An injective model $j : X \to \mathscr{L}(X)$ is an injective fibrant model since all anodyne extensions are trivial cofibrations for the injective structure and all injective objects are injective fibrant.

Thus, every weak equivalence (for the "new" model structure) is a local weak equivalence.

If $f : X \to Y$ is a local weak equivalence, then $\mathscr{L}(X) \to \mathscr{L}(Y)$ is a local weak equivalence between injective fibrant models, and is therefore a (standard) homotopy equivalence.

It follows that f is a weak equivalence in the "new" sense.

2) Motivic homotopy theory

Suppose that S is a scheme of finite dimension (typically a field), and let $(Sm|_S)_{Nis}$ be the category of smooth schemes of finite type over S, equipped with the Nisnevich topology.

Recall (from Lecture 02) that a covering family for the Nisnevich topology is an étale covering family $\phi_i : V_i \to U$ in the category of S-schemes such that every map $\mathrm{Sp}(K) \to U$ lifts to some V_i, for all fields K.

Nisnevich originally called this topology the "completely decomposed topology" or "cd-topology" [8], because of the way it behaves over fields — see [3].

The motivic model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ can be constructed in two ways:

a) Let S consist of the generating set of the trivial cofibrations for the injective model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$, plus the 0-section $* \to \mathbb{A}^1$, and let $I = \Delta^1$.

b) Let S be the generating set of trivial cofibrations for the injective model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ and let $I = \mathbb{A}^1$ with the global sections $0, 1 : * \to \mathbb{A}^1$.

It's an exercise to show that the two model structures coincide: show that every anodyne cofibration of one structure is a trivial cofibration of the other, and so the two structures have same injective objects.

It follows that the two classes of weak equivalences coincide.

The motivic model structure is called the $\mathbb{A}^1$-model structure in [7].

Strictly speaking, the Morel-Voevodsky model structure is on the category of simplicial sheaves on the smooth Nisnevich site, but the model structures for simplicial sheaves and simplicial presheaves are Quillen equivalent by the usual argument (Theorem 12.1).

There are many other models for motivic homotopy theory, including model structures on presheaves and sheaves (not simplicial!) on the smooth Nisnevich site [4], and all the models arising from test categories [5].

3) Localized model structures

Suppose that $f : A \to B$ is a cofibration of simplicial presheaves on a site $\mathscr{C}$.

Let S consist of the generating set of trivial cofibrations for the injective model structure on $s\mathrm{Pre}(\mathscr{C})$, plus the cofibration f. Let $I = \Delta^1$.

The resulting model structure is the f-local model structure on $s\mathrm{Pre}(\mathscr{C})$.

The motivic model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ is a special case of this construction, as are all of the standard f-local theories for simplicial sets.

4) Quasi-categories

The quasi-category model structure on the category $s\mathbf{Set}$ of simplicial sets is the model structure given by the theorem for the set S of **inner anodyne extensions**

$$\Lambda^n_k \subset \Delta^n,\ 0 < k < n,$$

and the interval $I = B\pi(\Delta^1)$.

Theorem 22.14. *Suppose that $f : * \to A$ is a global section of a simplicial presheaf A on a small site $\mathscr{C}$.*

Then the f-local model structure on $s\mathrm{Pre}(\mathscr{C})$ is proper.

Proof. We only need to verify right properness.

According to Theorem 22.3, and since the standard injective model structure on $s\mathrm{Pre}(\mathscr{C})$ is proper, it is enough to show that the map f_* is a weak equivalence in all pullback diagrams

$$\begin{tikzcd} F \arrow[r, "f_*"] \arrow[d] & A \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ * \arrow[r, "f"'] & A \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

such that p is a fibration and Y is fibrant.

The map $t : A \to *$ is a weak equivalence and Y is fibrant, so there is a map $v : * \to Y$ and a homotopy h making the diagram

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d, "0"'] & A \arrow[d, "0"] \arrow[drr, "\alpha"] & & \\ \Delta^1 \arrow[r] & A \times \Delta^1 \arrow[rr, "h"] & & Y \\ * \arrow[r, "f"'] \arrow[u, "1"] & A \arrow[r, "t"'] \arrow[u, "1"'] & * \arrow[ur, "v"'] & \end{tikzcd}$$

commute.

All instances of the maps 0 and 1 pull back to weak equivalences along p since the standard injective model structure is proper.

It therefore suffices to show that the map f_* in the pullback diagram

$$\begin{array}{ccccc} F_v & \xrightarrow{f_*} & A \times F_v & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ * & \xrightarrow[f]{} & A & \xrightarrow[vt]{} & Y \end{array}$$

is a weak equivalence, where F_v is the fibre of p over v, but this is obvious since f_* is anodyne. □

Corollary 22.15. *The motivic model structure on the category* $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ *of simplicial presheaves on the smooth Nisnevich site is proper.*

References

[1] Denis-Charles Cisinski. Les préfaisceaux comme modèles des types d'homotopie. *Astérisque*, 308:xxiv+390, 2006.

[2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[3] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[4] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[5] J. F. Jardine. Categorical homotopy theory. *Homology, Homotopy Appl.*, 8(1):71–144 (electronic), 2006.

[6] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.

[7] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, 90:45–143 (2001), 1999.

[8] Ye. A. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In *Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987)*, volume 279 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 241–342. Kluwer Acad. Publ., Dordrecht, 1989.