Joins, slices, (co)limits in higher category theory

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The goal of this talk is to extend the join and slice constructions to the higher-categorical context and to use these to define (co)limits of diagrams indexed by simplicial sets in a quasi-category. The main reference is [Joy02].

Definition. The *join* of simplicial sets is the unique functor *: sSet \times sSet \rightarrow sSet together with natural transformations $K \rightarrow K * L \leftarrow L$ such that:

- 1. $\Delta^m * \Delta^n \cong \Delta^{m+n+1}$
- 2. The functors $*L : sSet \rightarrow L/sSet$ and $K * : sSet \rightarrow K/sSet$ preserve colimits.

Explicitly, we can describe the *n*-simplices of A * B as follows:

$$(A*B)_n = \prod_{i+j=n} A_i \times B_j$$

for $n \ge 0$, with the convention that $A_{-1} = B_{-1} = \{\bullet\}$. Notice that $A * \emptyset = \emptyset * A = A$.

This join generalizes the join of 1-categories: if A, B are 1-categories we have a natural isomorphism $N(A * B) \simeq N(A) * N(B)$.

Proposition 1. *If C and D are quasicategories then C* * *D is a quasicategory.*

Proof. Consider an inner horn in the join $p : \Lambda_i^n \to C * D$, where 0 < i < n. If p factors through one of the inclusions $C \hookrightarrow C * D \leftrightarrow D$ we are done. So we may suppose that p carries the vertices $\{0, \ldots, j\}$ into C and the vertices $\{j + 1, \ldots, n\}$ into D. Restricting p we get maps $\Delta^{\{0, \ldots, j\}} \to C$, $\Delta^{\{j+1, \ldots, n\}} \to D$ which together determine a map $\Delta^n \simeq \Delta^{\{0, \ldots, j\}} * \Delta^{\{j+1, \ldots, n\}} \to C * D$ that solves the lifting problem.

Proposition 2. For any simplicial set T the functor

$$-*T: \mathsf{sSet} \to T/\mathsf{sSet}$$

 $S \mapsto (T \hookrightarrow S * T)$

has a right adjoint.

Proof. Immediate by the Adjoint Functor Theorem.

We will write X/t (or X/T if there is no risk of confusion) for the value of right adjoint $T/sSet \rightarrow sSet$ on $t : T \rightarrow X$. By definition, we have:

$$\hom_t(S * T, X) \simeq \hom(S, X/t)$$

where a map $\hom_t(S * T, X)$ is the set of simplicial maps $f : S * T \to X$ such that $f|_T = t$. This means in particular that we can describe the *n*-simplices of the slice X/t as:

$$(X/t)_n = \{ y : \Delta^n * T \to X \mid Y|_T = t \}.$$

The following are useful identities between joins of simplices, horns and spheres.

Lemma 3. We have identifications:

- 1. $(\partial \Delta^m * \Delta^n) \cup (\Delta^m * \partial \Delta^n) \simeq \partial \Delta^{m+n+1}.$
- 2. $(\Lambda_k^m * \Delta^n) \cup (\Delta^m * \partial \Delta^n) \simeq \Lambda_k^{m+n+1}$.
- 3. $(\partial \Delta^m * \Delta^n) \cup (\Delta^m * \Lambda^n_k) \simeq \Lambda^{m+n+1}_{m+k+1}$

The theory that we will describe depends on the following technical lemma.

Lemma 4. Given $i : A \hookrightarrow B$ and $j : S \hookrightarrow T$ inclusions of simplicial sets, consider the inclusions $u : (A * T) \cup (B * S) \hookrightarrow B * T$. Fix a diagram $t : T \to X$ and define $s := t \circ j : S \to X$. For any simplicial map $f : X \to Y$ there is a canonical bijection between the class of commutative squares of the form:



in T/sSet, and the class of commutative squares of the form:



in sSet.

Moreover a square of one class has a diagonal filler if and only if its corresponding square in the other class has a diagonal filler.

We will see that a slice over a quasicategory is a quasicategory, which follows from the fact that the projection from a slice simplicial set $X/t \rightarrow X$ is a right fibration. But actually a much more general statement holds (Theorem 6). We first show this for simplices.

Lemma 5. Let $f : X \to Y$ be an inner fibration and $t : \Delta^n \to X$ a simplicial map. If $0 \le k < n$ the projection:

$$p: X/\Delta^n \to (X/\Lambda^n_k) \times_{Y/\Lambda^n_k} (Y/\Delta^n)$$

is a trivial fibration.

Proof. By Lemma 4 it suffices to show that:



has a diagonal filler. Using (the dual of) the second identity in Lemma 3, u becomes the inclusion $\Lambda_{k+m+1}^{m+n+1} \hookrightarrow \Delta^{m+n+1}$. So the square has a diagonal filler since f is an inner fibrations and 0 < m+k+1 < m+n+1.

Theorem 6. Let $f : X \to Y$ be an inner fibration, $j : S \hookrightarrow T$ an inclusion, $t : T \to X$ a simplicial map, and $s := t \circ j$. Then the projection:

$$p: X/t \to (X/ti) \times_{Y/fs} (Y/ft)$$

is a right fibration.

Proof. By Lemma 4, it suffices to show that every commutative square of the form:



has a diagonal filler. By the dual of Lemma 4, this is equivalent to showing that every commutative square of the form:



has a diagonal filler. By the dual of Lemma 5, q is a trivial fibration since $0 < k \le n$. The map $S \hookrightarrow T$ is a cofibration hence this last square has diagonal filler as needed.

Corollary 7. Given an inclusion $j : S \hookrightarrow T$, set $s := t \circ j : T \to C$, then the induced projection $C/t \to C/s$ is a right fibration. In particular, if C is a quasicategory then so is the slice C/t for any map $t : T \to C$.

Proof. Setting $Y = \Delta^0$ in Theorem 6 we deduce the first statement. Setting $S = \emptyset$ and C = X the right fibration p is the projection $C/t \to C$, and since C is a quasicategory C/t is also a quasicategory.

Corollary 8. Let $f : C \to D$ be a map between quasicategories and $S \hookrightarrow T$ an inclusion of simplicial sets. Then the simplicial set $(C/S) \times_{D/S} (D/T)$ is a quasciategory and the projection $p_1 : (C/S) \times_{D/S} (D/T) \to C/S$ is a right fibration.

Proof. Consider the pullback square:



Then *q* is a right fibration by Corollary 7. This implies that p_1 is a right fibration, which proves the second statement. Now again by Corollary 7, *C*/*S* is a quasicategory so $C/S \times_{D/S} D/T$ is a quasicategory, since p_1 is an inner fibration.

We are ready to discuss cones and limits.

Definition 9. Let *K* be a simplicial set and *C* a quasicategory.

- (i) A *diagram* in *C* indexed by *K* is a simplicial map $X : K \to C$.
- (ii) A *cone* over X is a simplicial map $\overline{X} : K^{\triangleleft} \to C$ such that $\overline{X}|_{K} = X$, which means that the following diagram commutes:



(iii) A *universal cone* over X is a cone $\overline{X} : K^{\triangleleft} \to C$ with the property that for all n > 0 and all $Z : \partial \Delta^n * K \to C$ such that $Z|_{\Delta^{\{n\}}*K} = \overline{X}$ there exists an extension:



Example 10.

- (i) A *terminal object* in a quasicategory *C* is a limit of the diagram $\emptyset \to C$.
- (ii) A *pullback* in a quasicategory *C* is a limit of a diagram $\Lambda_2^2 \rightarrow C$.

The space of cones over a fixed diagram $X : K \to C$ can be identified with the quasicategory C/X. Let $(C/K)^{\text{univ}}$ be the simplicial set spanned by the universal cones in C/X. Then, by adjunction, the condition for universality of the cones translates to:



Corollary 11. In the context above, the simplicial set $(C/K)^{univ}$ is either empty or a contractible Kan complex. In other words, the limit of a diagram, if it exists, it is unique up to a contractible space of choices.

Let us finish by proving some expected properties of the notions introduced above.

Proposition 12. *Given a quasicategory* C *and a diagram* $X : K \to C$ *, the limits of this diagram are in correspondence with the terminal objects in* C/X*.*

Proof. A cone $\overline{X} : K^{\triangleleft} \to C$ corresponds by adjunction to a map $\overline{X} : \Delta^0 = \Delta^0 * \emptyset \to C/X$, which is a cone over the empty diagram. Using again the adjunction, universality of $\overline{X} : K^{\triangleleft} \to C$ as a limit of X corresponds precisely to universality of $\overline{X} : \Delta^0 * \emptyset \to C/X$ as a limit of the empty diagram.

Proposition 13. *Given a quasicategory* C *and a vertex* x *the identity map* Id_x *is terminal in* C/x*.*

To prove this we need a theorem from Karol's talk. Recall that an outer horn $x : \Lambda_n^n \to X$ (with n > 1) in a simplicial set X is *special* if the edge $x | \Delta^{\{n,n+1\}}$ is an equivalence.

Theorem 14. In a quasicategory any special horn can be filled.

Proof of Proposition 13. Given a lifting problem:

$$\begin{array}{c} \partial \Delta^n \xrightarrow{Z} C/x \\ & & \\ & & \\ & & \\ & & \\ \Delta^n \end{array}$$

with $Z|\Delta^{\{n\}} = \mathsf{Id}_x$ we consider its adjoint problem:

$$\begin{array}{ccc} \partial \Delta^n \ast \Delta^0 & \xrightarrow{Z} & C \\ & & & & \\ & & & & \\ & & & & \\ \Delta^n \ast \Delta^0 \end{array}$$

Notice that the vertical map is the inclusion $\Lambda_{n+1}^{n+1} \hookrightarrow \Delta^{n+1}$, then by the above theorem it is enough to show that *Z* is a special horn. But by construction we have $Z|\Delta^{\{n,n+1\}} = Id_x$, so we are done.

References

[Joy02] André Joyal. Quasi-categories and kan complexes. *Journal of Pure and Applied Algebra*, 175(1):207–222, 2002.