The Yoneda embedding for quasicategories

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The goal of this talk is to introduce two different ways in which the Yoneda embedding can be constructed in the higher categorical setting. Our references are:

- [Lur09] for the construction involving simplicially enriched categories.
- [Joy08] and [Joy09] for the construction using the universal left fibration.

To be able to define the Yoneda embedding we first need to find a suitable replacement for the category of sets and for the notion of a presheaf. The former is given by the *quasicategory of spaces* S, defined as the homotopy coherent nerve N(Kan) of the simplicial category Kan of Kan complexes. Consequently, by a *presheaf* on a quasicategory C we will understand a simplicial map $C^{op} \to S$, and the quasicategory of presheaves is the exponential $\mathcal{P}(C) := S^{C^{op}}$. Since S is a quasicategory this is again a quasicategory.

We want to construct a simplicial map $C \to \mathcal{P}(C)$. By definition of $\mathcal{P}(C)$ and the exponential law this would be the same as having a map $C^{op} \times C \to S$. Since $S = \mathbf{N}(Kan)$, by the adjunction $\mathfrak{C} \dashv \mathbf{N}$ it is enough to construct a simplicial functor $\mathfrak{C}(C^{op} \times C) \to Kan$. To get this consider the map $u : \mathfrak{C}(C^{op} \times C) \to \mathfrak{C}(C^{op}) \times \mathfrak{C}(C)$ given by the universal property of the product. This map can be composed with

$$\mathfrak{C}(K)^{op} \times \mathfrak{C}(K) \to \mathsf{Kan}$$

 $x, y \mapsto (\hom_{\mathfrak{C}(K)}(x, y))^{\sim}$

where the tilde indicates that we are taking a fibrant replacement in the Quillen model structure (for example we can use Ex^{∞} or the singular complex of the geometric realization of $\hom_{\mathfrak{C}(K)}(x, y)$). This gets us a map $Y_C : \mathfrak{C}(C^{op} \times C) \to \mathsf{Kan}$.

Definition 1 (Lurie). By the exponential law and the adjunction $\mathfrak{C} \dashv \mathbf{N}$ the functor Y_C corresponds to a simplicial map $\mathcal{Y}_C : C \to \mathcal{P}(C)$, the *Yoneda embedding* of *C*.

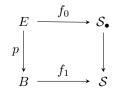
Note that the above construction would be greatly simplified if \mathfrak{C} preserved products, but it's a simple exercise to show that it doesn't. We have the expected statement.

Proposition 2 ([Lur09, 5.1.3.1]). *For any simplicial set* K *the map* $\mathcal{Y}_K : K \to \mathcal{P}(K)$ *is fully faithful.*

We now give a more conceptual approach to the theory of presheaves in the context of quasicategories. Consider the functor $Q : sSet^{op} \rightarrow Cat$ that maps a simplicial set B to the homotopy category of the covariant model structure on sSet /B. Then Q is (almost) representable by S in the following sense (the only reason why it is not really representable is because S is not small).

Proposition 3. The category Q(B) is naturally equivalent to the homotopy category of the quasicategory S^B .

Let us explain the previous proposition in more detail. Recall that the fibrant objects in the covariant model structure on sSet /B are exactly left fibrations. The proposition asserts that there exists a universal left fibration over S. The total space of this fibration is given by *the quasicategory of pointed spaces* defined as the slice quasicategory $S_{\bullet} := 1/S$. The universality in this context means that for any left fibration $E \rightarrow B$ there exists a homotopy pullback in sSet_J:



where (f_0, f_1) are unique, in the sense that the Kan complex of such maps is contractible. The map f_1 is called the *classifying map* of the left fibration p.

We use the proposition to define the Yoneda embedding $C \to \mathcal{P}(C)$: we first construct a left fibration over $C^{op} \times C$ and then consider the classifying map $C^{op} \times C \to S$.

Definition 4. For any simplicial set define the *twisted diagonal* as the simplicial set C^{δ} with *n*-simplices given by:

$$(C^{\delta})_n := \hom_{\mathsf{sSet}}((\Delta^n)^{op} * \Delta^n, C).$$

Example 5. Notice that the twisted diagonal of the nerve of a 1-category C is the nerve of the category of elements of the hom functor $\hom_{C} : C^{op} \times C \to Set$.

Notice that the twisted diagonal comes with a natural projection $(s,t) : C^{\delta} \to C^{op} \times C$ induced by the inclusions $(\Delta^n)^{op} \hookrightarrow (\Delta^n)^{op} * \Delta^n \leftrightarrow \Delta^n$. By [Joy08, 14.23], the projection (s,t)is a left fibration, so we can consider its classifying map that we denote by $\hom_C : C^{op} \times C \to S$.

Definition 6 (Joyal). The *Yoneda embedding* is the transpose of the classifying map $\hom_C : C^{op} \times C \to S$.

The equivalence between the two definitions of the Yoneda embedding is proven in [Lur12, 5.2.1.11].

References

- [Joy08] André Joyal. Notes on quasi-categories. unpublished manuscript, 2008.
- [Joy09] André Joyal. The theory of quasi-categories and its applications. *unpublished manuscript*, 2009.

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- [Lur12] Jacob Lurie. Higher algebra. avaible from author's website, 2012.