EQUIVALENT NOTIONS OF ∞-TOPOI

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Abstract. This talk introduces ∞-topoi and Giraud's axiomatic characterization of them. The ∞-categorical generalizations of sheaves on a site will also be discussed.

Notations

We will use the following notations and terminology.

- By an ∞-*category*, we mean a quasi-category.
- $\mathcal{S}$ is the (large) ∞-category given by the homotopy coherent nerve of the simplicial category Kan of *small* Kan complexes.
- $\widehat{\mathcal{S}}$ is the (very large) ∞-category given by the homotopy coherent nerve of the simplicial category KAN of *all* Kan complexes.
- Cat_∞ is the (large) ∞-category given by the homotopy coherent nerve of the simplicial category $\mathsf{QCat}^{\text{core}}$ having *small* quasi-categories as objects and mapping spaces given by $\text{core}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ for small ∞-categories $\mathcal{C}, \mathcal{D}$. Here, for $\mathcal{C}$ an ∞-category, $\text{core}(\mathcal{C})$ is the maximal sub-Kan complex of $\mathcal{C}$.
- $\widehat{\mathsf{Cat}_\infty}$ is the (very large) ∞-category given by the homotopy coherent nerve of the simplicial category $\mathsf{QCAT}^{\text{core}}$ having *all* quasi-categories as objects and defined analogously to $\mathsf{QCat}^{\text{core}}$.
- For a simplicial set X, $\mathsf{PrSh}(X)$ is the ∞-category $\text{Fun}(X^{\text{op}}, \mathcal{S})$ of functors $X^{\text{op}} \to \mathcal{S}$.
- For an ∞-category $\mathcal{C}$, $\mathbf{y}\colon \mathcal{C} \to \mathsf{PrSh}(\mathcal{C})$ denotes the Yoneda embedding.

1. Preliminaries

1.1. **Ordinary Giraud's Theorem.** We start from the 1-categorical characterization of (Grothendieck) topoi.

Theorem 1.1 ([SGA4], Exposé IV, Thm 1.2)**.** *For an ordinary category $\mathcal{X}$, the following are equivalent.*

(1) $\mathcal{X}$ is equivalent to $\mathsf{Sh}(\mathcal{C})$, the category of sheaves on a small Grothendieck site $\mathcal{C}$.

(2) $\mathcal{X}$ is a left exact localization of $\mathsf{PrSh}(\mathcal{C}) = \mathsf{Set}^{\mathcal{C}^{\text{op}}}$, for a small category $\mathcal{C}$, i.e. $\mathcal{X}$ is equivalent to a (full, replete) reflective subcategory of $\mathsf{PrSh}(\mathcal{C})$ for which the left adjoint to the inclusion functor preserves finite limits.

(3) $\mathcal{X}$ satisfies Giraud's axiom*:*

(a) *$\mathcal{X}$ is a locally presentable category;*

(b) *colimits in $\mathcal{X}$ are universal;*

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(c) *coproducts in $\mathcal{X}$ are disjoint;*
(d) *every equivalence relation in $\mathcal{X}$ is effective.*

We do not spell out the meaning of conditions (b)-(d) above, since we will redefine them in the ∞-categorical setting.

Remark 1.2. Our goal is to illustrate to what extent the above result generalizes to ∞-categories. We will see that, in the context of ∞-categories, one only has

$$(1) \implies (2) \iff (3)$$

Indeed, in order to obtain the equivalence between (1) and (2), we will have to restrict to a subclass of left exact localization of $\mathsf{PrSh}(\mathcal{C})$ (for an ∞-category $\mathcal{C}$) – the *topological* localizations.

1.2. **Locally presentable ∞-categories and adjoints.** We recall some preliminary results on locally presentable ∞-categories.

Theorem 1.3 ([Lur09], Prop 5.5.2.2)**.** *Let $\mathcal{X}$ be a locally presentable ∞-category and let $F \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{S}$ be a presheaf on $\mathcal{X}$. Then F is representable if and only if it preserves small limits.*

Theorem 1.4 ([Lur09], Cor 5.5.2.9)**.** (Adjoint Functor Theorem). *Let $F \colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a functor between locally presentable ∞-categories. Then:*

(1) F is a left adjoint if and only if it preserves small colimits;
(2) F is a right adjoint if and only if it is accessible and preserves small limits.

The above result makes the following notations sensible.

Notation 1.5. We define subcategories $\mathsf{Pr}^{\mathsf{L}}, \mathsf{Pr}^{\mathsf{R}} \subseteq \widehat{\mathsf{Cat}}_\infty$ as follows:

- the objects of both Pr^{L} and Pr^{R} are the locally presentable ∞-categories;
- the morphisms in Pr^{L} are the functors between ∞-categories preserving small colimits;
- the morphisms in Pr^{R} are the functors between ∞-categories preserving small limits.

1.3. **Truncated objects.** Recall that a Kan complex X is (-2)-*truncated* if it is contractible, whereas it is k-*truncated*, for an integer $k \geq -1$, if $\pi_i(X, x)$ is trivial for all $i > k$ and all vertices x of X. A map $f \colon X \to Y$ is k-truncated, for $k \geq -2$, if all of its homotopy fibers are k-truncated.

Definition 1.6. Let $\mathcal{C}$ be an ∞-category and $k \geq -2$ an integer.

(1) An object C of $\mathcal{C}$ is k-*truncated* if, for every object D of $\mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(D, C)$ is k-*truncated*.
(2) A morphism $f \colon C \to D$ of $\mathcal{C}$ is k-*truncated* if, for all objects Z of $\mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(f, Z)$ is k-truncated.
(3) A morphism $f \colon C \to D$ of $\mathcal{C}$ is a *monomorphism* if it is (-1)-truncated.

Remark 1.7. A morphism $f \colon C \to D$ is k-truncated in $\mathcal{C}$ if and only if it is k-truncated when seen as an object of $\mathcal{C}/D$.

If $\mathcal{X}$ is a locally presentable ∞-category and X is an object in $\mathcal{X}$, we let $\mathrm{Sub}(X)$ be the collection of isomorphism classes of objects for $\tau_{\leq -1}(\mathcal{X}/X)$ – the full subcategory of $\mathcal{X}/X$ spanned by the monomorphisms in $\mathcal{X}$ with codomain X. Then $\mathrm{Sub}(X)$ is a (small) poset (see [Lur09], Prop 6.2.1.4).

2. Grothendieck topologies on ∞-categories

We start by generalizing Grothendieck sites and sheaves on them to the setting of ∞-categories.

Definition 2.1. Let $\mathcal{C}$ be an ∞-category.

(1) A *sieve* on $\mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ such that, if D is an object of $\mathcal{C}^{(0)}$ and there is a morphism $f : C \to D$ in $\mathcal{C}$, then C is also in $\mathcal{C}^{(0)}$.

(2) Let C be an object of $\mathcal{C}$. A *sieve on* C is a sieve on $\mathcal{C}/C$.

(3) Let $(\mathcal{C}/C)^{(0)}$ be a sieve on the object C and $f : D \to C$ be a morphism in $\mathcal{C}$. We let $f^*(\mathcal{C}/C)^{(0)}$ be the full subcategory of $\mathcal{C}/D$ spanned by those $g : D' \to D$ in $\mathcal{C}$ such that fg is equivalent to an object of $(\mathcal{C}/C)^{(0)}$.

Definition 2.2. (1) A *Grothendieck topology* on an ∞-category $\mathcal{C}$ is an assignment, for every object C of $\mathcal{C}$, of a collection of sieves on C – the *covering sieves* – such that:

(i) $\mathcal{C}/C$ is a covering sieve, for every object C of $\mathcal{C}$;

(ii) if $(\mathcal{C}/C)^{(0)}$ is a covering sieve on C and $f : D \to C$ is a morphism in $\mathcal{C}$, $f^*(\mathcal{C}/C)^{(0)}$ is a covering sieve of D;

(iii) if $(\mathcal{C}/C)^{(0)}$ is a covering sieve on C and $(\mathcal{C}/C)^{(1)}$ is any sieve on C such that, for all $f : D \to C$ in $(\mathcal{C}/C)^{(0)}$, $f^*(\mathcal{C}/C)^{(0)}$ is a covering sieve on D, then $(\mathcal{C}/C)^{(1)}$ is a covering sieve.

(2) A *Grothendieck site* is an ∞-category $\mathcal{C}$ equipped with a Grothendieck topology.

Note that, when $\mathcal{C}$ is the nerve of an ordinary category, the above Definition coincides with the usual one for 1-categories. In effect, more generally we have the following

Remark 2.3. For an ∞-category $\mathcal{C}$, there is a bijection between the collection of Grothendieck topologies on $\mathcal{C}$ and the collection of Grothendieck topologies on $\mathrm{Ho}(\mathcal{C})$ (see [Lur09], Remark 6.2.2.3).

As in ordinary category theory, sieves on objects of an ∞-category coincide with subobjects of the associated representable presheaf.

Proposition 2.4 ([Lur09], Prop 6.2.2.5)**.** *Let $\mathcal{C}$ be an ∞-category and $\mathbf{y} : \mathcal{C} \to \mathsf{PrSh}(\mathcal{C})$ the Yoneda embedding. For an object C in $\mathcal{C}$ and a monomorphism $i : U \to \mathbf{y}(C)$, let $(\mathcal{C}/C)(i)$ be the full subcategory of $\mathcal{C}/C$ given by those morphisms $f : D \to C$ such that $\mathbf{y}(f)$ factors through i. Then the assignment $i \mapsto (\mathcal{C}/C)(i)$ gives a bijection*

$$\mathrm{Sub}(\mathbf{y}(C)) \longleftrightarrow \{\text{sieves on } C\}$$

With the above characterization in hand, the ∞-categorical generalization of ordinary sheaves is straightforward.

Definition 2.5. Let $\mathcal{C}$ be a (small) Grothendieck site and let S be the collection of all monomorphisms $U \to \mathbf{y}(\mathcal{C})$ corresponding to covering sieves in $\mathcal{C}$. A presheaf $F \in \mathsf{PrSh}(\mathcal{C})$ is a $\mathcal{C}$-*sheaf* (or just a *sheaf* for short) if it is S-local. The full subcategory of $\mathsf{PrSh}(\mathcal{C})$ spanned by the sheaves is denoted by $\mathsf{Sh}(\mathcal{C})$.

By definition, $\mathsf{Sh}(\mathcal{C})$ is a localization of $\mathsf{PrSh}(\mathcal{C})$, hence we obtain a localization functor

$$L : \mathsf{PrSh}(\mathcal{C}) \to \mathsf{Sh}(\mathcal{C}),$$

given by the left adjoint to the inclusion $\mathsf{Sh}(\mathcal{C}) \subseteq \mathsf{PrSh}(\mathcal{C})$. Localizations $\mathsf{PrSh}(\mathcal{C})_S$ of $\mathsf{PrSh}(\mathcal{C})$ that are sheaves on a Grothendieck sites can be characterized in terms of a certain property of the collection S of morphisms in $\mathsf{PrSh}(\mathcal{C})$. Recall first the following

Definition 2.6 ([Lur09], Def 5.5.4.5)**.** Let $\mathcal{X}$ be a cocomplete ∞-category. A collection S of morphisms in $\mathcal{X}$ is *strongly saturated* if:

(1) pushouts of morphisms in S along arbitrary morphisms of $\mathcal{X}$ belong to S;

(2) the full subcategory of $\mathrm{Fun}(\Delta[1], \mathcal{X})$ spanned by S is closed under small colimits;

(3) S has the two-out-of-three property.

Definition 2.7. Let $\mathcal{X}$ be an ∞-category with finite limits.

(1) A strongly saturated class of morphisms $\overline{S}$ in $\mathcal{X}$ is called *left exact* if pullbacks of morphisms in $\overline{S}$ along arbitrary morphisms of $\mathcal{X}$ belong to $\overline{S}$.
(2) If $\mathcal{X}$ is a locally presentable ∞-category, a left exact collection $\overline{S}$ of morphisms in $\mathcal{X}$ is called *topological* if there is a set S of monomorphisms in $\mathcal{X}$ such that $\overline{S}$ is the smallest strongly saturated class containing S.
(3) A localization functor $F\colon \mathcal{X} \to \mathcal{Y}$ is called a *left exact localization* (resp. a *topological localization*) if the class of all morphisms f in $\mathcal{X}$ such that Lf is an equivalence in $\mathcal{Y}$ is left exact (resp. topological).

Proposition 2.8 ([Lur09], Prop 6.2.1.1)**.** *Let $\mathcal{X}$ be a category with finite limits. Then a functor $F\colon \mathcal{X} \to \mathcal{Y}$ is a left exact localization if and only if it preserves finite limits.*

Here is the result we were looking for.

Theorem 2.9 ([Lur09], Lemma 6.2.2.7 & Prop 6.2.2.9)**.** *(1) Let $\mathcal{C}$ be a (small) Grothendieck site. Then $L\colon \mathsf{PrSh}(\mathcal{C}) \to \mathsf{Sh}(\mathcal{C})$ is a topological localization.*
(2) There is a bijection between Grothendieck topologies on $\mathcal{C}$ and equivalence classes of topological localizations of $\mathsf{PrSh}(\mathcal{C})$.

In general, we give the following

Definition 2.10. A locally presentable category is an ∞-*topos* if there is a left exact localization functor $L\colon \mathsf{PrSh}(\mathcal{C}) \to \mathcal{X}$, for some small ∞-category $\mathcal{C}$.

Thus, Theorem 2.9 says that the ∞-topoi which are sheaves on a Grothendieck sites are exactly those for which there exists a *topological* localization $L\colon \mathsf{PrSh}(\mathcal{C}) \to \mathcal{X}$.

3. Codomain fibration and universal colimits

Definition 3.1. Let $\mathcal{X}$ be an ∞-category. The functor

$$\mathrm{cod}_{\mathcal{X}}\colon \mathrm{Fun}(\Delta[1], \mathcal{X}) \to \mathcal{X}$$

induced by the right anodyne map $\{1\} \subseteq \Delta[1]$ is called the *codomain fibration*.

The usage of the term "fibration" is justified by the following

Proposition 3.2. *For an ∞-category $\mathcal{X}$, $\mathrm{cod}_{\mathcal{X}}$ is a coCartesian fibration.*

Proof. [Lur09], Cor 2.4.7.12 applied to the identity functor on $\mathcal{X}$ gives that

$$\mathrm{Fun}(\{0\} \subseteq \mathcal{X})\colon \mathrm{Fun}(\Delta[1], \mathcal{X}) \to \mathrm{Fun}(\{0\}, \mathcal{X})$$

is a Cartesian fibration. Taking duals we conclude. □

As a coCartesian fibration, $\mathrm{cod}_{\mathcal{X}}$ is coclassified, by the dual of the straightening-unstraightening construction that we saw in Aji's talk, by the functor

$$\mathcal{X} \to \widehat{\mathsf{Cat}}_\infty$$
$$X \mapsto \mathcal{X}/X, \qquad (f\colon X \to Y) \mapsto (f_!\colon \mathcal{X}/X \to \mathcal{X}/Y),$$

where $f_!$ can be thought of as postcomposition with f.

Proposition 3.3 ([Lur09], Lemma 6.1.1.1)**.** *Let* $\mathcal{X}$ *be an* ∞*-category and let*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

be a commutative square in $\mathcal{X}$*, corresponding to a functor* $\Delta[1] \to \mathrm{Fun}(\Delta[1], \mathcal{X})$*. Such a square is a* $\mathrm{cod}_{\mathcal{X}}$*-Cartesian edge if and only if it is a pullback square in* $\mathcal{X}$*.*

Corollary 3.4. *Let* $\mathcal{X}$ *be an* ∞*-category admitting pullbacks. Then* $\mathrm{cod}_{\mathcal{X}}$ *is a Cartesian fibration classified by*

$$\mathcal{X}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$$
$$X \mapsto \mathcal{X}/X, \qquad (f\colon X \to Y) \mapsto (f^*\colon \mathcal{X}/Y \to \mathcal{X}/X),$$

where f^* *can be thought of as pullback along* f*.*

Remark 3.5. If $\mathcal{X}$ has pullbacks, we get, for all morphisms $f\colon X \to Y$, a pair of adjoint functors:

$$\mathcal{X}/X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{X}/Y \quad (f_! \dashv f^*)$$

Definition 3.6. Let $\mathcal{X}$ be a cocomplete ∞-category with pullbacks. We say that *colimits in* $\mathcal{X}$ *are universal* if, for any morphism $f\colon X \to Y$, $f^*\colon \mathcal{X}/Y \to \mathcal{X}/X$ preserves small colimits.

Remark 3.7 ([Lur09], Prop 6.1.1.4)**.** By Theorem 1.4, in a locally presentable ∞-category $\mathcal{X}$ colimits are universal if and only if the functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$ classifying the Cartesian fibration $\mathrm{cod}_{\mathcal{X}}$ factors through Pr^{L}.

We record here a little consequence of universality of colimits that we will need later.

Lemma 3.8. *Let* $\mathcal{X}$ *be a locally presentable* ∞*-category in which colimits are universal. If* X *is an object of* $\mathcal{X}$ *and there is a morphism* $f\colon X \to \emptyset$ *in* $\mathcal{X}$*, then* X *is initial.*

Proof. The object $\mathrm{id}_\emptyset$ is both initial and terminal in $\mathcal{X}/\emptyset$. By hypothesis, $f^*\colon \mathcal{X}/\emptyset \to \mathcal{X}/X$ preserves both limits and colimits, so $f^*(\mathrm{id}_\emptyset)$ is both initial and terminal in $\mathcal{X}/X$. It follows that id_X, being another terminal object in X/X, is also initial. This means that X is initial in $\mathcal{X}$, by [Lur09], Prop 1.2.13.8. ☐

Before continuing with the formulation of the ∞-categorical version of Giraud's axioms, we describe how pushouts and pullbacks diagrams interact in an ∞-topos. We start with the following

Definition 3.9. Let X be an ∞-category and K a simplicial set. A natural transformation $\alpha\colon p \to q$ between functors $p, q\colon K \to \mathcal{X}$ is *Cartesian* if, for every 1-simplex $\varphi\colon x \to y$ in K, the induced diagram

$$\begin{array}{ccc} p(x) & \xrightarrow{p(\varphi)} & p(y) \\ {\scriptstyle \alpha(x)}\downarrow & & \downarrow{\scriptstyle \alpha(y)} \\ q(x) & \xrightarrow[q(\varphi)]{} & q(y) \end{array}$$

is a Cartesian square in $\mathcal{X}$.

Let now S be a collection of morphisms in an ∞-category $\mathcal{X}$ and denote by $\mathcal{X}^S$ the full subcategory of $\mathrm{Fun}(\Delta[1], \mathcal{X})$ spanned by S. If $\mathcal{X}$ has pullbacks, we let $\mathsf{Cart}_{\mathcal{X}}$ be the subcategory of $\mathrm{Fun}(\Delta[1], \mathcal{X})$ having the same objects of $\mathrm{Fun}(\Delta[1], X)$ but with morphisms given by Cartesian squares in $\mathcal{X}$. Note that, if S is *stable under pullback* (i.e. pullbacks of morphisms in S along arbitrary morphisms of $\mathcal{X}$ are still in S), we have

$$\mathsf{Cart}_{\mathcal{X}}^S = \mathsf{Cart}_{\mathcal{X}} \cap \mathcal{X}^S.$$

Furthermore, by [Lur09], Cor 2.4.2.5, $\mathrm{cod}_{\mathcal{X}}$ restricts to a Cartesian fibration $\mathcal{X}^S \to \mathcal{X}$ and to a right fibration $\mathsf{Cart}_{\mathcal{X}}^S \to \mathcal{X}$.

Proposition 3.10 ([Lur09], Lemma 6.1.3.7)**.** *Let $\mathcal{X}$ be a locally presentable ∞-category in which colimits are universal and let S be a class of morphisms in $\mathcal{X}$ which is stable under pullbacks. The following are equivalent:*

(1) the Cartesian fibration $\mathrm{cod}_{\mathcal{X}}\colon \mathcal{X}^S \to \mathcal{X}$ *is classified by a colimit-preserving functor* $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$*;*

(2) the right fibration $\mathrm{cod}_{\mathcal{X}}\colon \mathsf{Cart}_{\mathcal{X}}^S \to \mathcal{X}$ *is classified by a colimit-preserving functor* $\mathcal{X}^{\mathrm{op}} \to \mathcal{S}$*;*

(3) S is stable under small coproducts and, for every pushout diagram

$$\begin{array}{ccc} f & \xrightarrow{\alpha} & g \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ f' & \xrightarrow[\alpha]{} & g' \end{array}$$

in $\mathrm{Fun}(\Delta[1], \mathcal{X})$*, if α, β are Cartesian and $f, f', g \in S$, then α', β' are also Cartesian and $g' \in S$.*

Definition 3.11. Let S be a class of morphisms in a locally presentable ∞-category $\mathcal{X}$ in which colimits are universal. We say that S is *local* if it is stable under pullbacks and satisfies one of the equivalent conditions of Proposition 3.10.

Proposition 3.12 ([Lur09], Thm 6.1.3.9)**.** *For a locally presentable ∞-category $\mathcal{X}$, the following are equivalent:*

(1) colimits in $\mathcal{X}$ are universal and the collection of all morphisms in $\mathcal{X}$ is local;

(2) the Cartesian fibration $\mathrm{cod}_{\mathcal{X}}\colon \mathrm{Fun}(\Delta[1], \mathcal{X}) \to \mathcal{X}$ *is classified by a limit-preserving functor* $\mathcal{X}^{\mathrm{op}} \to \mathsf{Pr}^{\mathsf{L}}$*.*

Proposition 3.13 ([Lur09], Prop 6.1.3.10)**.** *If $\mathcal{X}$ is an ∞-topos, then colimits in $\mathcal{X}$ are universal and the collection $\mathcal{X}_1$ of* all *morphisms in $\mathcal{X}$ is local.*

4. Giraud's Axioms

Definition 4.1. Let $\mathcal{X}$ be an ∞-category with finite coproducts. We say that *coproducts in $\mathcal{X}$ are disjoint* if, for all objects X, Y of $\mathcal{X}$, the pushout diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \coprod Y \end{array}$$

is also a pullback.

For ∞-categories, the properties required by Giraud's axioms of relations being effective is substituted with the same sort of requirement for groupoid objects.

Definition 4.2. Let $\mathcal{X}$ be an ∞-category. A *groupoid object* in $\mathcal{X}$ is a simplicial object $G\colon \Delta^{\mathrm{op}} \to \mathcal{X}$ with the following property. For all $n \in \mathbb{N}$ and all $I, J \subseteq [n]$ such that $I \cup J = [n]$ and $I \cap J = \{i\}$, the square

$$\begin{array}{ccc} G([n]) & \longrightarrow & G(I) \\ \downarrow & & \downarrow \\ G(J) & \longrightarrow & G(\{i\}) \end{array}$$

is Cartesian in $\mathcal{X}$.

We let Δ_+ be the category of finite ordinals and set $[-1] := \emptyset \in \Delta_+$. Observe that $\Delta_+ \cong \Delta[0] * \Delta$. A functor $\Delta_+ \to \mathcal{X}$ is called an *augmented simplicial object* in the $\infty-$category $\mathcal{X}$.

Definition 4.3. An augmented simplicial object $U\colon \Delta_+ \to \mathcal{X}$ in an ∞-category $\mathcal{X}$ is a *Čech nerve* if the restriction of U along $\Delta \subseteq \Delta_+$ is a groupoid object in $\mathcal{X}$ and

$$\begin{array}{ccc} U_1 & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & U_{(-1)} \end{array}$$

is a Cartesian square in $\mathcal{X}$.

Note that, up to equivalence, a Čech nerve is determined by the map $u\colon U_0 \to U_{(-1)}$.

Definition 4.4. Let $\mathcal{X}$ be an ∞-category and let $G\colon \Delta^{\mathrm{op}} \to \mathcal{X}$ be a groupoid object in $\mathcal{X}$.

(1) A colimit diagram of G is denoted by $|G|\colon \Delta_+^{\mathrm{op}} \to \mathcal{X}$ and called a *geometric realization* of G.

(2) The groupoid G is called *effective* if $|G|$ is a Čech nerve.

We can then finally give the following

Definition 4.5. Let $\mathcal{X}$ be an ∞-category. We say that $\mathcal{X}$ *satisfies Giraud's axioms* if

(a) $\mathcal{X}$ is locally presentable;

(b) colimits in $\mathcal{X}$ are universal;

(c) coproducts in $\mathcal{X}$ are disjoint;

(d) every groupoid object in $\mathcal{X}$ is effective.

Proposition 4.6 ([Lur09], Prop 6.1.3.19)**.** *Let X be a locally presentable ∞-category verifying one of the equivalent conditions of Proposition 3.12. Then $\mathcal{X}$ satisfies Giraud's axioms.*

Proof. Axioms (a) and (b) are given by hypothesis. We will only show that coproducts in $\mathcal{X}$ are disjoint. Consider X, Y objects in $\mathcal{X}$ and let $f\colon \emptyset \to X$. Form the following pushout diagram in $\mathrm{Fun}(\Delta[1], \mathcal{X})$

$$\begin{array}{ccc} \mathrm{id}_\emptyset & \xrightarrow{\alpha} & \mathrm{id}_Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ f & \xrightarrow[\alpha']{} & g \end{array}$$

Here $\alpha = (\emptyset \to Y,\ \emptyset \to Y)$ is clearly Cartesian and the same is true of $\beta = (\mathrm{id}_\emptyset, f)$ thanks to Lemma 3.8. By Proposition 3.12, we can then deduce that α' is Cartesian and, by applying the codomain fibration to the above pushout diagram, α' can be identified with the pushout square

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \coprod Y \end{array}$$

which is therefore also a pullback. □

The above Proposition together with Proposition 3.13 implies that every ∞-topos satisfies Giraud's axioms. In effect, we get the

Theorem 4.7 ([Lur09], Prop 6.1.5.3)**.** *For an ∞-category $\mathcal{X}$, the following are equivalent:*

(1) $\mathcal{X}$ is an ∞-topos;
(2) $\mathcal{X}$ satisfies Giraud's axioms.

5. Small Object Classifier

We conclude with a discussion about object classifiers in an ∞-topos.

Definition 5.1. Let $\mathcal{X}$ be an ∞-category admitting pullbacks and let S be a collection of morphisms in $\mathcal{X}$ which is stable under pullbacks.

(1) A morphism $\pi\colon X \to Y$ in $\mathcal{X}$ is said to *classify* S if it is a final object in $\mathsf{Cart}^S_{\mathcal{X}}$. If this holds, we also say that Y is a *classifying object* for S and that π is the *universal morphism with property* S.
(2) If S is the collection of all monomorphisms in $\mathcal{X}$, a classifying object for S is called a *subobject classifier* for $\mathcal{X}$.

Remark 5.2. The universality of the morphism $\pi\colon X \to Y$ classifying S is explained by the existence of a zig-zag of trivial fibrations

$$\mathsf{Cart}^S_{\mathcal{X}} \xleftarrow{\ \sim\ } (\mathsf{Cart}^S_{\mathcal{X}})/\pi \xrightarrow{\ \sim\ } \mathcal{X}/Y$$

The leftmost trivial fibration is just the statement that π is initial in $\mathsf{Cart}^S_{\mathcal{X}}$, whereas the rightmost one is saying that every $Y' \to Y$ (seen as an object in $\mathcal{X}/Y$) can be lifted to a Cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \longrightarrow & Y \end{array}$$

where the left vertical map is in S.

Definition 5.3. Let $\mathcal{X}$ be an ∞-category and κ an uncountable regular cardinal. We say that $\mathcal{X}$ is *essentially κ-small* if it is a κ-compact object in Cat_∞. We say $\mathcal{X}$ is *essentially small* if it is essentially κ-small for some uncountable regular cardinal κ.

Here is a sufficient and necessary criterion for the existence of a classifying object for a collection S of morphisms in a locally presentable ∞-category.

Proposition 5.4 ([Lur09], Prop 6.1.6.3)**.** *Let $\mathcal{X}$ be a locally presentable ∞-category in which colimits are universal. Let S be a class of morphisms in $\mathcal{X}$ which is stable under pullback. Then there is a classifying morphism for S if and only if:*

(i) *S is local;*

and

(ii) *for all objects X in $\mathcal{X}$, $\mathcal{X}/X$ is essentially small.*

Proof. If $S^\sharp\colon \mathcal{X}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ classifies $\mathrm{cod}_{\mathcal{X}}\colon \mathsf{Cart}_{\mathcal{X}}^{S} \to \mathcal{X}$ as a Cartesian fibration, then there is a classifying morphism for S if and only if $S^\sharp$ is representable. This, in turn, is equivalent to $S^\sharp$ preserving small limits and factoring through $\mathcal{S}$, i.e. to (i) and (ii) being verified. □

Definition 5.5. Let κ be an uncountable regular cardinal. A morphism $f\colon X \to Y$ in $\mathcal{X}$ is *relatively κ-compact* if, for all pullback squares

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

whenever Y' is κ-small, then so is X'.

Proposition 5.6 ([Lur09], Prop 6.1.6.7)**.** *Let $\mathcal{X}$ be a locally presentable ∞-category in which colimits are universal and suppose S is a local class of morphisms in $\mathcal{X}$. For an uncountable regular cardinal κ, let S_κ denote the subclass of relatively κ-compact morphisms in S. Then, for κ large enough, S_κ has a classifying morphism.*

The above two Propositions imply the following characterizations of ∞-topoi in terms of classifying objects.

Theorem 5.7 ([Lur09], Thm 6.1.6.8)**.** *The following are equivalent, for a locally presentable ∞-category $\mathcal{X}$:*

(1) $\mathcal{X}$ is an ∞-topos;

(2) colimits in $\mathcal{X}$ are universal and, for all sufficiently large regular cardinals κ, there is a classifying object in $\mathcal{X}$ for the class of relatively κ-compact maps in $\mathcal{X}$.

References

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[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009, http://dx.doi.org/10.1515/9781400830558 doi:10.1515/9781400830558.