(Co)cartesian families in simplicial type theory

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Draft article: https://www2.mathematik.tu-darmstadt.de/~buchholtz/fib-syn.pdf

1. Simplicial Type Theory
2. (Co)-Cartesian Families
3. A 2-Yoneda Lemma
4. Outlook

Feel free to interrupt!

Why simplicial type theory?

Open problem: Can we define & develop the theory of $(\infty, 1)$-categories in (book) HoTT? Can we define the type of semi-simplicial types?

- If we can, it'll likely be a rather complicated construction, and it will be useful to have a DSL (domain specific language) in order to reason practically with $(\infty, 1)$-categories.
- If we can't, it'll still be nice to have a synthetic type theory (DSL) to use until we settle on the proper extension of HoTT. (Maybe *Two-level type theory*?)

A DSL: Simplicial type theory (Riehl–Shulman: *A type theory for synthetic ∞-categories*)

Related work: Harper–Licata, Warren, Nuyts, Licata–Weaver, Cavallo–Riehl–Sattler, Riehl–Verity, Cisinski, North, ...

∞-cosmos

Why (co)cartesian families?

RS defined covariant and contravariant families, representing copresheaves and presheaves over a base category B, *i.e.*, co-/contravariant functors $P : B \to \mathrm{Space}$.

Here, we study (co)cartesian families, representing co-/contravariant functors $P : B \to \mathrm{Cat}$.

These can model the higher-categorical versions of $\mathrm{Mod} : \mathrm{Ring} \to \mathrm{Cat}$ and $\mathrm{Vect} : \mathrm{Mfld} \to \mathrm{Cat}$, for example.

Another use: *symmetric monoidal* $(\infty, 1)$*-categories* are cocartesian families over the category of finite pointed sets, Fin_*.

They are also a crucial stepping stone toward *defining* the universe Cat itself.

Outline

Simplicial Type Theory

The simplest simplicial type theory: Postulate:

$$\begin{tikzcd}
\Delta^1 = \{x = y\} \arrow[r] \arrow[d] & \{x \leq y\} = \Delta^2 \arrow[d] \\
\Delta^2 = \{y \leq x\} \arrow[r] & \{x \leq y \vee y \leq x\} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$$\begin{aligned}
\mathbb{2} &: \mathrm{Set} \\
\leq &: \mathbb{2} \to \mathbb{2} \to \mathrm{Prop} \\
0, 1 &: \mathbb{2} \\
p &: \text{“}(\mathbb{2}, \leq, 0, 1) \text{ is a strict interval”}
\end{aligned}$$

$\mathbb{2} = \Delta^1$

$0 \longrightarrow 1$

A *strict interval* is a totally ordered set with distinct top and bottom elements.

Indeed, the 1-topos of simplicial sets is the classifying 1-topos for the theory of strict intervals.

In particular, the square $\mathbb{2} \times \mathbb{2}$ is obtained by gluing together two 2-simplices $\Delta^2 :\equiv \{(x, y) : \mathbb{2} \times \mathbb{2} \mid y \leq x\}$.

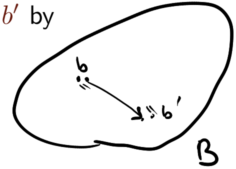
As a consequence, we can define connection maps $\wedge, \vee : \mathbb{2} \times \mathbb{2} \to \mathbb{2}$.

More shapes; hom-types

We can (uniformly) define the simplices $\Delta^n = \{(x_1, \ldots, x_n) : \mathbb{2}^n \mid 0 \leq x_n \leq \cdots \leq x_1 \leq 1\}$, the horns Λ^n_k and the boundaries $\partial\Delta^n$, along with the embeddings $\Lambda^n_k \hookrightarrow \partial\Delta^n \hookrightarrow \Delta^n$.

Given a type B with elements $b, b' : B$, we define the type of arrows from b to b' by

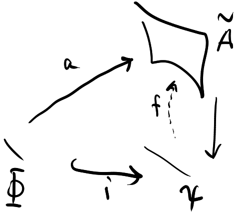
$$(b \to_B b') :\equiv \hom_B(b, b') :\equiv \sum_{u : \mathbb{2} \to B} (u\,0 = b) \times (u\,1 = b').$$



More generally, we can introduce the *extension type* as an abbreviation for extensions, given $i : \Phi \hookrightarrow \Psi$, $A : \Psi \to \mathcal{U}$, $a : \prod_{x:\Phi} A(i\,x)$:

$$\left\langle \prod_{x:\Psi} A(x) \Big|^{\Phi}_{a} \right\rangle :\equiv \sum_{f : \prod_{x:\Psi} A(x)} (a = f \circ i)$$

$\simeq$



(This is a primitive type former in RS, using the shape+tope machinery.)

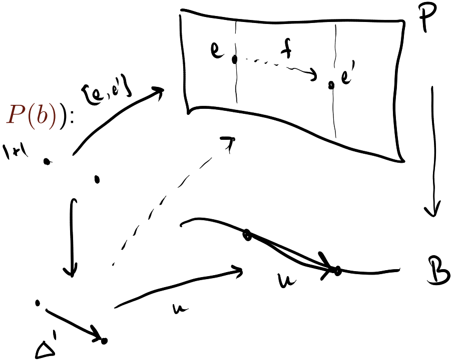
Dependent arrows

Given a type family $P : B \to \mathcal{U}$ and an arrow $u : \hom_B(b, b')$ in the base, and elements $e : P(b)$ and $e' : P(b')$, the type of arrows from e to e' over f is defined by:

$$(e \to_f^P e') :\equiv \sum_{f : \prod_{t:\mathbb{2}} P(u\,t)} (f\,0 =_{u_0}^P e) \times (f\,1 =_{u_1}^P e')$$

This is simply the type of lifts in the square ($E :\equiv \sum_{b:B} P(b)$):

$$\begin{tikzcd}
1+1 \arrow[r, "{[e,e']}"] \arrow[d, "{[0,1]}"'] & E \arrow[d, "\pi"] \\
\mathbb{2} \arrow[r, "u"'] & B
\end{tikzcd}$$



Segal types

$$
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[d, "1"'] & \Delta^1 \arrow[d] \\
\Delta^1 \arrow[r] & \Lambda^2_1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

The *Segal* types are the local types wrt the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$. That is, B is Segal if the restriction map

$$(\Delta^2 \to B) \to (\Lambda^2_1 \to B)$$

is an equivalence. The Segal types form a reflective subuniverse.

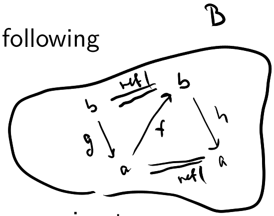
The Segal types in simplicial spaces model pre-$(\infty, 1)$-categories, or equivalently, flagged $(\infty, 1)$-categories.

(!)

Associativity follows. *Question:* In this setting, can we derive the uniform Segal condition, *i.e.*, locality wrt to the spine inclusions $\mathrm{Sp}^n \hookrightarrow \Delta^n$, for all $n : \mathbb{N}$?

Isomorphisms

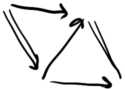
An arrow $f : a \to b$ in a Segal type B is a (categorical) *isomorphism* if the following proposition(!) holds:



$$\mathrm{isIso}(f) :\equiv \sum_{g:b\to a} \sum_{h:b\to a} (hf = \mathrm{id}_a) \times (fg = \mathrm{id}_b).$$

The type of isomorphisms $a \simeq_B b :\equiv \sum_{f:a\to_B b} \mathrm{isIso}(f)$ is equivalent to the mapping type $\mathbb{E} \to B$, where $\mathbb{E}$ is the colimit of the diagram:

$$\begin{tikzcd}
 & \Delta^1 \arrow[dl] \arrow[dr, "d^1"] & & \Delta^1 \arrow[dl, "d^0"'] \arrow[dr, "d^2"] & & \Delta^1 \arrow[dl, "d^1"'] \arrow[dr] & \\
\mathbf{1} & & \Delta^2 & & \Delta^2 & & \mathbf{1}
\end{tikzcd}$$



Rezk types

Fix a Segal type B. Then B is a *Rezk* type iff it is $\mathbb{E}$-null, *i.e.*, the map $B \to (\mathbb{E} \to B)$ is an equivalence.

Equivalently, B is $(k : \mathbf{1} \to \mathbb{E})$-local, for either $k = 0, 1$.

Rezk types are our internal $(\infty, 1)$-categories. (Univalent pre-$(\infty, 1)$-categories, flagged $(\infty, 1)$-categories where the flag contracts away.)

A type is *discrete* if it is Δ^1-null. Discrete types are Rezk, and model ∞-groupoids.

The Yoneda Lemma

A family/map $\pi : E \to B$ is *covariant* (*contravariant*) if it is right orthogonal to $0 : \mathbf{1} \hookrightarrow \Delta^1$ ($1 : \mathbf{1} \hookrightarrow \Delta^1$).

Yoneda Lemma (RS)

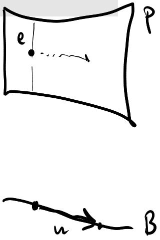
If B is Segal, $b : B$, and $P : B \to \mathcal{U}$ is covariant, then evaluation gives an equivalence:

$$\left(\hom(b,-) \to P\right) = \left(\prod_{x:B} (b \to_B x) \to P(x)\right) \to P(b).$$

Dependent version: If B Segal, $b : B$, $P : b/B \to \mathcal{U}$ covariant, then evaluation gives equivalence:

$$\left(\prod_{x:B} \prod_{f:b \to_B x} P(x,f)\right) \to P(b, \mathrm{id}_b)$$

dir. path induction



Directed encode-decode

Remark We have the following analog of the fundamental theorem of identity types:

Observation

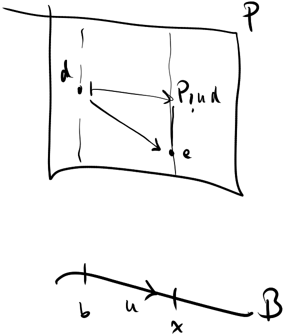
Let B be a Segal type, $b : B$, and let $P : B \to \mathcal{U}$ be a covariant family with $d : P(b)$. The fiberwise map $\prod_{x:B}((b \to_B x) \to P(x))$ given by covariance, is a fiberwise equivalence if and only if $\langle b, d\rangle$ is initial in $\sum_{x:B} P(x)$. φ

$\Longrightarrow$ $\langle b, \mathrm{id}_b \rangle$ is initial in $\Sigma_{x:B} (b \to_B x)$

$\Longleftarrow$ $\mathrm{fib}\, \varphi_x\, e = \sum_{(u : b \to_B x)} (P_! u\, d = e)$

$$\simeq \sum_{(u : b \to_B x)} d \xrightarrow[u]{P} e \simeq \langle b, d\rangle \xrightarrow[\Sigma_B P]{} \langle x, e \rangle$$

$$\simeq \mathbb{1}.$$



Outline

Inner and Isoinner Families

We say that a map $\pi : E \to B$ is *inner* if it is right orthogonal to the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$.

Note that if B is Segal, then E is Segal iff π is inner.

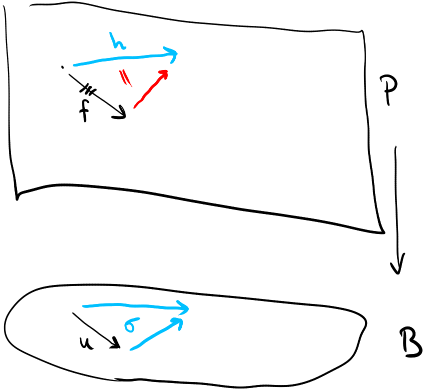
We say that π is *isoinner* if it is in addition $\mathbb{E}$-null.

If B is Rezk, then E is Rezk iff π is isoinner.

(Co)-Cartesian Arrows

Let B be a type and $P : B \to \mathcal{U}$ be an inner family. Let $b, b' : B$, $u : b \to_B b'$, and $e : P\,b$, $e' : P\,b'$. An arrow $f : e \to^P_u e'$ is a P-*cocartesian arrow* if and only if the following proposition holds:

$$\mathrm{isCocartArr}_P f :\equiv \prod_{\sigma : \left\langle \Delta^2 \to B \middle|^{\Delta^1_0}_{u} \right\rangle} \; \prod_{h : \prod_{t:\Delta^1} P\,\sigma(t,t)} \mathrm{isContr}\left(\left\langle \prod_{\langle t,s\rangle : \Delta^2} P\sigma(t,s) \middle|^{\Lambda^2_0}_{[f,h]} \right\rangle \right).$$



We say that f is a P-*cocartesian lift* if u starting at e.

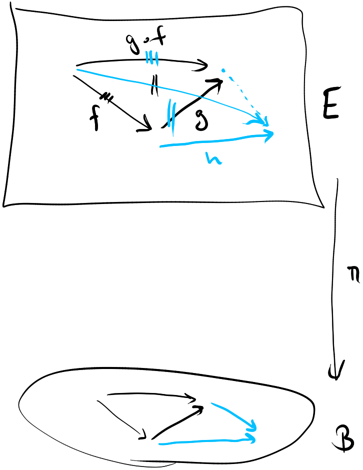
Lemma If B is Rezk and P is isoinner, then P-cocartesian lifts are unique. (The type of them is a proposition.)

A cancellation property

Let $\pi : E \to B$ be a map of Rezk types.
If f, g are composable arrows in E, and f is cocartesian,
then g is cocartesian iff $g \circ f$ is.

$\Rightarrow$ easy!

$\Leftarrow$



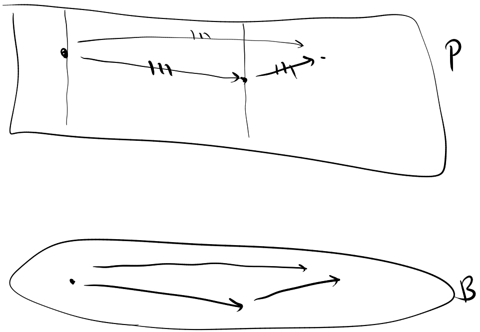
(Co)-Cartesian Families

Let B be Rezk. We say that the isoinner family $P : B \to \mathcal{U}$ is *cocartesian* if all cocartesian lifts exists. (This is a proposition.)

Taking the right endpoint of the lifts gives functoriality maps

$$P_! : (b \to_B b') \to P(b) \to P(b')$$

compatible with composition.



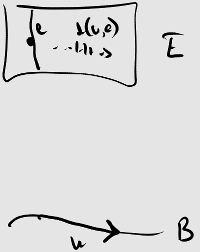
The Chevalley criterion

left adjoint right inverse

Theorem

A map $\pi : E \to B$ *of Rezk types is cocartesian iff we have a LARI adjunction in:*

$$\begin{tikzcd}
E^{\Delta^1} \arrow[dr, bend left, "r"] \arrow[drr, bend left=30, "\partial_0"] \arrow[ddr, bend right, "\pi^{\Delta^1}"'] & & \\
& \pi \downarrow B \arrow[ul, dotted, bend left, "\ell"] \arrow[ul, phantom, "\top"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\
& B^{\Delta^1} \arrow[r, "\partial_0"] & B
\end{tikzcd}$$



Similarly, we give a fibered adjunction criterion.

Corollary (Co)cartesian maps are closed under dependent product (hence exponentiation), composition and pullback. The domain map $\partial_0 : B^{\Delta^1} \to B$ is cartesian and is cocartesian if B has pushouts; dually, the codomain map $\partial_1 : B^{\Delta^1} \to B$ is cocartesian and is cartesian if B has pullbacks.

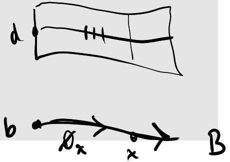
Outline

A 2-Yoneda Lemma

Yoneda Lemma (Dependent version)

Let B be a Rezk type with initial object b, and let $P : B \to \mathcal{U}$ be cocartesian. Then evaluation at b induces an equivalence

$$\mathrm{ev}_b : \left(\prod_{x:B}^{\mathrm{cocart}} P(x) \right) \to P(b)$$



Here we take the subtype of *cocartesian sections*, *i.e.*, those mapping arrows to cocartesian arrows.

Corollary

Let B be a Rezk type, $b : B$ any element, and $Q : B \to \mathcal{U}$ cocartesian. Then evaluation at id_b gives an equivalence:

$$\left\{ \begin{tikzcd} b/B \arrow[rr, dashed, "\mathrm{cocart}"] \arrow[dr] & & \tilde{2} \arrow[dl] \\ & B & \end{tikzcd} \right\} = \left(b/B \to_B^{\mathrm{cocart}} Q\right) \simeq \left(\prod_{u:b/B}^{\mathrm{cocart}} Q(u\,1) \right) \to Q(b)$$

$$b/B \xrightarrow{\partial_1} B \to Q$$

" hom(b, −) corr. fam "

A 2-Yoneda Lemma, Proof

Define <u>LARI</u> $\mathbf{y} : P(b) \to \prod^{\mathrm{cocart}}_{x:B} P(x)$ as follows:

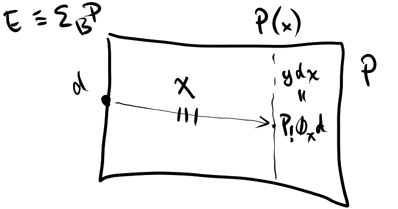
$$
\begin{tikzcd}
\prod^{\mathrm{cocart}}_B P \arrow[r, shift right, "\simeq"'] \arrow[d, "\int"] & P(b) \arrow[l, shift right, "y"'] \arrow[dl, bend left, "y"] \\
\prod_B P \arrow[ur, bend left, "\mathrm{ev}_b"] \arrow[ur, phantom, "\top"] &
\end{tikzcd}
$$

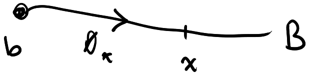
$y\,d\,x :\equiv P_!\,\emptyset_x\,d$

$d : P(b)$

$\mathrm{cst}_d : B \longrightarrow E$

$\chi : \mathrm{cst}_d \Rightarrow y\,d$





Let $u : x \underset{B}{\longrightarrow} x'$ be given.

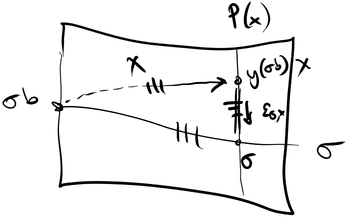
$$
\begin{tikzcd}
d \arrow[r, dashed, "\chi"] \arrow[d, equal] & y\,d\,x \arrow[d, ":\equiv\; y\,d\,(u)"] \\
d \arrow[r] & y\,d\,x'
\end{tikzcd}
\quad \checkmark
$$

A 2-Yoneda Lemma, Proof continued

Recap : have $ev_b : T :\equiv \prod_B^{cocart} P \longrightarrow P(b) : y$

$ev_b \circ y = id_{P(b)}$

$$\left(y \circ ev_b = id_T \right) \simeq \prod_{\sigma : T} \prod_{x:B} \left(\overset{\varepsilon_{\sigma,x}}{y(\sigma b) \simeq \sigma x} \right)$$

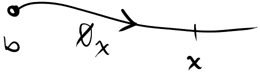


$\square$

$\varepsilon_{\sigma,x}$ is the filler from χ to $\sigma(\varnothing_x)$

this is a vertical cocart. arrow $\therefore$ iso

$\therefore$ id.



Outline

Outlook

$$\begin{tikzcd}
\mathcal{E}^{\Delta^1} \arrow[d, shift right=4.5] \arrow[d, shift left=1.5] \\
\mathcal{E} \arrow[u, shift right=1.5] \arrow[u, shift left=4.5]
\end{tikzcd}$$

- Bring in cohesion: Free co-/contravariant families, flat (Conduché) maps, descent?
- Bring in cubical exolayer: Universes Space and Cat, univalent?
- Bring in more modalities, op and tw. The naïve Yoneda lemmas.
- ...
- What structure/axioms would be sufficient to get a foundational system, not just a DSL?

Δ^1 is tiny

$P : B^{op} \to \mathrm{Space}$

$\mathrm{hom} : B^{op} \to B \to \mathcal{U}$