UNIVERSITETET I BERGEN
Det matematisk-naturvitenskapelige fakultet

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Symmetries of $\mathbb{S}^n$

in univalent foundations

Homotopy Type Theory Electronic Seminar Talks
November 19th, 2020

*At the time

1. Symmetries of the circle

2. Symmetries of the 2-sphere

3. Symmetries of higher spheres

1. Symmetries of the circle

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such that $f(\bullet) \equiv t$ and $[f](\circlearrowleft) = \ell$.

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- the point $\bullet : \mathbb{S}^1$,
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Goal: prove that

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$$(\mathbb{S}^1 \simeq \mathbb{S}^1) \simeq (\mathbb{S}^1 \longrightarrow \mathbb{S}^1)_{(\mathrm{id}_{\mathbb{S}^1})} + (\mathbb{S}^1 \longrightarrow \mathbb{S}^1)_{(-\mathrm{id}_{\mathbb{S}^1})}$$

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- Identify both component with $\mathbb{S}^1$.

$\mathrm{id}_{\mathbb{S}^1} \neq -\mathrm{id}_{\mathbb{S}^1}$

Suppose $p : \mathrm{id}_{\mathbb{S}^1} = -\mathrm{id}_{\mathbb{S}^1}$, and evaluate:

$$p(\bullet) : \bullet = \bullet$$

$$[p](\circlearrowleft) : p(\bullet) =_{\circlearrowleft} p(\bullet)$$

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Suppose $p : \mathrm{id}_{\mathbb{S}^1} = -\mathrm{id}_{\mathbb{S}^1}$, and evaluate:

$$\simeq \mathbb{Z}$$

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i.e.

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Consequence: $\left\| f = \mathrm{id}_{\mathbb{S}^1} \right\| + \left\| f = -\mathrm{id}_{\mathbb{S}^1} \right\|$ is a proposition for $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

Equivalences are merely $\mathbf{id}_{\mathbb{S}^1}$ or $-\mathbf{id}_{\mathbb{S}^1}$

Let $\phi : \mathbb{S}^1 \simeq \mathbb{S}^1$ and prove the proposition $\left\| \phi = \mathrm{id}_{\mathbb{S}^1} \right\| + \left\| \phi = -\mathrm{id}_{\mathbb{S}^1} \right\|$.

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$$\begin{tikzcd} \bullet \arrow[r, shift left, "p"] & \phi(\bullet) \arrow[l, shift left, "p^{-1}"] \arrow[loop right, "{[\phi](\circlearrowleft)}"] \end{tikzcd}$$

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Because ϕ equivalence: $p^{-1}\phi(\circlearrowleft)p = \circlearrowleft^{\pm 1}$.

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In other words there is $e_1 : \phi = \mathrm{id}_{\mathbb{S}^1}$ or $e_{-1} : \phi = -\mathrm{id}_{\mathbb{S}^1}$. Then truncate.

$\mathbb{S}^1 \longrightarrow \mathbb{S}^1$

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\mathrm{id}_{\mathbb{S}^1} \longmapsto\ & (\bullet, 1) \\
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\end{aligned}
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Conclusion: $(\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{S}^1 + \mathbb{S}^1$

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 & \boxed{(\mathbb{S}^1 \times \mathbb{Z})_{((\bullet,1))}} \arrow[u] & & \boxed{(\mathbb{S}^1 \times \mathbb{Z})_{((\bullet,-1))}} \arrow[u]
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& \boxed{\mathbb{S}^1} \arrow[u] & & \boxed{\mathbb{S}^1} \arrow[u]
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2. Symmetries of the 2-sphere

$\mathbb{S}^2$ as a suspension

$\mathbb{S}^2 :\equiv \Sigma\mathbb{S}^1$ is the suspension of $\mathbb{S}^1$, defined by:

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such that $f(N) \equiv n$, $f(S) \equiv s$ and $[f] \circ \mathrm{mrd} = m$.

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Probably not: the argument for $\mathbb{S}^1 \xrightarrow{\simeq} (\mathbb{S}^1 \longrightarrow \mathbb{S}^1)_{(\mathrm{id}_{\mathbb{S}^1})}$ relies on

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Still plausible: there is two equivalent connected components, one at $\mathrm{id}_{\mathbb{S}^2}$, the other at $-\mathrm{id}_{\mathbb{S}^2}$.

Definition of $-\mathbf{id}_{\mathbb{S}^2}$

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N &\longmapsto n \ :\equiv S \\
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Hopefully, $\mathrm{id}_{\mathbb{S}^2} \neq -\mathrm{id}_{\mathbb{S}^2}$ still holds.

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WLOG, one can suppose $p(N) = \mathbf{mrd}(\bullet)$ and $p(S) = \mathbf{mrd}(\bullet)^{-1}$ inhabited.

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Then one has

$$\pi : \prod_{x : \mathbb{S}^1} \mathrm{mrd}(\bullet) =_{\mathrm{mrd}(x)} \mathrm{mrd}(\bullet)^{-1}$$

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... ultimately, one gets an element of $[\mathrm{mrd}](\circlearrowleft^2) = \mathrm{refl}_{\mathrm{mrd}(\bullet)}$

$\mathbf{id}_{\mathbb{S}^2} \neq -\mathbf{id}_{\mathbb{S}^2}$ (cont'd)

Recall the Hopf family:

$$\mathcal{H} : \mathbb{S}^2 \longrightarrow \mathcal{U}$$

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Recall the Hopf family:

$$\begin{aligned}\mathcal{H} \,:\, \mathbb{S}^2 &\longrightarrow \mathcal{U} \\ N &\mapsto n :\equiv \mathbb{S}^1 \\ S &\mapsto s :\equiv \mathbb{S}^1 \\ x &\mapsto (x, \circlearrowleft_x) \;:\; \mathbb{S}^1 \xrightarrow{m} (\mathbb{S}^1 = \mathbb{S}^1)\end{aligned}$$

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Then $[\mathcal{H}] \circ \mathrm{mrd}$ is an equivalence, in particular $[[\mathcal{H}] \circ \mathrm{mrd}]$ is injective and one ends up with $\circlearrowleft^2 = \mathsf{refl}_\bullet$.

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Degree is a monoid morphism

The degree of a function $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ pointed by $f_0 : N = f(N)$ is

$$\pi_2(\mathbb{S}^2) \xrightarrow{\pi_2(f, f_0)} \pi_2(\mathbb{S}^2)$$

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Because π_2 is a functor:

$$d(\mathrm{id}_{\mathbb{S}^2}, \mathrm{refl}_N) = 1$$

$$d(\ (g, g_0) \circ (f, f_0)\ ) = d(g, g_0) \times d(f, f_0)$$

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Consequence: the degree maps pointed equivalences to either 1 or -1.

Alternative description

$$(\mathbb{S}^2 \to_* \mathbb{S}^2) \longrightarrow (\mathbb{S}^1 \to_* \Omega\,\mathbb{S}^2) \longrightarrow \Omega^2\,\mathbb{S}^2 \longrightarrow \Omega\,\mathbb{S}^1 \simeq \mathbb{Z}$$

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Define $\tau(p) :\equiv [\mathcal{H}](p)(\bullet)$ for $p : N = N$.

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$\mathbf{0}$-connected

Define $\tau(p) :\equiv [\mathcal{H}](p)(\bullet)$ for $p : N = N$.

Consequence: for $(f, f_0), (g, g_0) : \mathbb{S}^2 \longrightarrow_* \mathbb{S}^2$,

$$d(f, f_0) = d(g, g_0) \iff \|(f, f_0) = (g, g_0)\|.$$

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Recall one has $\mathrm{id}_{\mathbb{S}^2} \neq -\mathrm{id}_{\mathbb{S}^2}$.

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Then $d(f, f_0) = \pm 1$. Also, $d(\mathrm{id}_{\mathbb{S}^2}, \mathsf{refl}_N) = 1$ and $d(-\mathrm{id}_{\mathbb{S}^2}, \mathrm{mrd}(\bullet)) = -1$.

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This yield $\|(f, f_0) = (\mathrm{id}_{\mathbb{S}^2}, \mathsf{refl}_N)\| + \|(f, f_0) = (-\mathrm{id}_{\mathbb{S}^2}, \mathrm{mrd}(\bullet))\|$. From which derives $\|f = \mathrm{id}_{\mathbb{S}^2}\| + \|f = -\mathrm{id}_{\mathbb{S}^2}\|$.

Equivalence of both components

Define $\Psi : (\mathbb{S}^2 \longrightarrow \mathbb{S}^2) \longrightarrow (\mathbb{S}^2 \longrightarrow \mathbb{S}^2)$ by mapping a map f to:

$$\Psi(f)(N) :\equiv f(S), \quad \Psi(f)(S) :\equiv f(N), \quad [\Psi(f)] \circ \mathrm{mrd} = [f] \circ \mathrm{mrd}(-)^{-1}$$

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Hence,

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Conclusion for $n = 2$

$$(\mathbb{S}^2 = \mathbb{S}^2) \simeq \mathbf{2} \times (\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})}$$

3. Symmetries of higher spheres

Freudenthal's theorem

Inductively, $\mathbb{S}^{n+1} :\equiv \Sigma\mathbb{S}^n$ with the appropriate elimination rule.

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Freudenthal's suspension theorem implies that

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Hence, $\Omega^n(\sigma) : \Omega^n(\mathbb{S}^n) \longrightarrow \Omega^{n+1}(\mathbb{S}^{n+1})$ is $(n-2)$-connected.

$\mathbf{0}$-connectedness

$$\begin{tikzcd}
(\mathbb{S}^n \to_* \mathbb{S}^n) \arrow[r, "\Sigma(-)"] \arrow[d, "\wr"] & (\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1}) \arrow[d, "\wr"] \\
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Then $\Sigma(-) : (\mathbb{S}^n \to_* \mathbb{S}^n) \to (\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1})$ is $(n-2)$-connected, hence $\mathbf{0}$-connected.

0-connectedness

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(\mathbb{S}^n \longrightarrow_* \mathbb{S}^n) \arrow[r, "\Sigma(-)"] \arrow[d, "\wr"] & (\mathbb{S}^{n+1} \longrightarrow_* \mathbb{S}^{n+1}) \arrow[d, "\wr"] \\
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Because $\mathbb{S}^n$ and $\mathbb{S}^{n+1}$ are 1-connected, the forgetful maps

$$(\mathbb{S}^n \longrightarrow_* \mathbb{S}^n) \longrightarrow (\mathbb{S}^n \longrightarrow \mathbb{S}^n), \quad (\mathbb{S}^{n+1} \longrightarrow_* \mathbb{S}^{n+1}) \longrightarrow (\mathbb{S}^{n+1} \longrightarrow \mathbb{S}^{n+1})$$

are 0-connected.

Induction

$$\begin{tikzcd}
\left\|\mathbb{S}^n \to_* \mathbb{S}^n\right\|_0 \arrow[r, "\|\Sigma(-)\|_0", "\simeq"'] \arrow[d, "\wr"] & \left\|\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1}\right\|_0 \arrow[d, "\wr"] \\
\left\|\mathbb{S}^n \to \mathbb{S}^n\right\|_0 \arrow[r, "\|\Sigma(-)\|_0"'] & \left\|\mathbb{S}^{n+1} \to \mathbb{S}^{n+1}\right\|_0
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\end{tikzcd}$$

There is an isomorphism of monoids

$$\left\| \mathbb{S}^n \to \mathbb{S}^n \right\|_0 \to \left\| \mathbb{S}^{n+1} \to \mathbb{S}^{n+1} \right\|_0$$

for all $n \geq 2$, and the type of invertible elements of $\left\| \mathbb{S}^n \to \mathbb{S}^n \right\|_0$ is equivalent to $\left\| \mathbb{S}^n = \mathbb{S}^n \right\|_0$.

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As $\left\|\mathbb{S}^2 = \mathbb{S}^2\right\|_0 \simeq \mathbf{2}$, by induction $\left\|\mathbb{S}^n = \mathbb{S}^n\right\|_0 \simeq \mathbf{2}$ for all $n \geq 2$.

On the shape of $(\mathbb{S}^n = \mathbb{S}^n)_{(\mathrm{id}_{\mathbb{S}^n})}$

$$(\mathbb{S}^n \simeq_* \mathbb{S}^n) \longrightarrow (\mathbb{S}^n \simeq \mathbb{S}^n) \longrightarrow \mathbb{S}^n$$

is a fiber sequence.

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Hence a long exact sequence:

$$\ldots \longrightarrow \pi_2(\mathbb{S}^n) \longrightarrow \pi_1(\mathbb{S}^n \simeq_* \mathbb{S}^n) \longrightarrow \pi_1(\mathbb{S}^n \simeq \mathbb{S}^n) \longrightarrow \pi_1(\mathbb{S}^n) \longrightarrow \ldots$$

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So for $n \geq 3$,

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So for $n \geq 3$,

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- $(\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{S}^1 + \mathbb{S}^1$
- $(\mathbb{S}^2 = \mathbb{S}^2) \simeq \mathbf{2} \times (\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})}$
- $\|\mathbb{S}^n = \mathbb{S}^n\|_0 \simeq \mathbf{2}$ for $n \geq 3$

Sum up

What we have proved:

- $(\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{S}^1 + \mathbb{S}^1$
- $(\mathbb{S}^2 = \mathbb{S}^2) \simeq \mathbf{2} \times (\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})}$
- $\|\mathbb{S}^n = \mathbb{S}^n\|_0 \simeq \mathbf{2}$ for $n \geq 3$
- $(\mathbb{S}^n = \mathbb{S}^n) \neq (\mathbb{S}^n + \mathbb{S}^n)$ for $n \geq 3$

Sum up

What we have proved:

- $(\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{S}^1 + \mathbb{S}^1$
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What about:

Sum up

What we have proved:

- $(\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{S}^1 + \mathbb{S}^1$
- $(\mathbb{S}^2 = \mathbb{S}^2) \simeq \mathbf{2} \times (\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})}$
- $\|\mathbb{S}^n = \mathbb{S}^n\|_0 \simeq \mathbf{2}$ for $n \geq 3$
- $(\mathbb{S}^n = \mathbb{S}^n) \neq (\mathbb{S}^n + \mathbb{S}^n)$ for $n \geq 3$

What about:

- $(\mathbb{S}^2 = \mathbb{S}^2) \neq (\mathbb{S}^2 + \mathbb{S}^2)$?

Sum up

What we have proved:

- $(\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{S}^1 + \mathbb{S}^1$
- $(\mathbb{S}^2 = \mathbb{S}^2) \simeq \mathbf{2} \times (\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})}$
- $\|\mathbb{S}^n = \mathbb{S}^n\|_0 \simeq \mathbf{2}$ for $n \geq 3$
- $(\mathbb{S}^n = \mathbb{S}^n) \neq (\mathbb{S}^n + \mathbb{S}^n)$ for $n \geq 3$

What about:

- $(\mathbb{S}^2 = \mathbb{S}^2) \neq (\mathbb{S}^2 + \mathbb{S}^2)$?
- $(\mathbb{S}^n = \mathbb{S}^n) \simeq (\mathbb{S}^n = \mathbb{S}^n)_{(\mathrm{id}_{\mathbb{S}^n})} + (\mathbb{S}^n = \mathbb{S}^n)_{(-\mathrm{id}_{\mathbb{S}^n})}$?
  (In other words, is $\mathrm{id}_{\mathbb{S}^n} \neq -\mathrm{id}_{\mathbb{S}^n}$).
  Then $(\mathbb{S}^n = \mathbb{S}^n) \simeq \mathbf{2} \times (\mathbb{S}^n = \mathbb{S}^n)_{(\mathrm{id}_{\mathbb{S}^n})}$

Thank you.