# Univalence of the universal coCartesian fibration

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- D.-C. Cisinski, *Higher Categories and Homotopical Algebra*, Cambridge studies in advanced mathematics, vol. 180, Cambridge University Press, 2019.
- H. K. Nguyen, *Covariant & Contravariant Homotopy Theories*, arXiv:1908.06879.
- H. K. Nguyen, *A note on coCartesian fibrations*, in preparation.
- D.-C. Cisinski and H. K. Nguyen, *Univalence of the universal coCartesian fibration*, in preparation.

In $\infty$-category theory, a fundamental tool is straightening/unstraightening:

$$\left\{ \text{coCartesian fibrations} \quad \begin{array}{c} X \\ {\scriptstyle p}\downarrow \\ A \end{array} \right\} \quad \cong \quad \left\{ \text{functors } A \xrightarrow{F} \infty\text{-}Cat \right\}$$

where $\infty$-$Cat$ denotes the $\infty$-category of small $\infty$-categories, and fibrations are required to have small fibers in the sense that, for any pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ A' \arrow[r] & A \end{tikzcd} \qquad A' \text{ small} \Rightarrow X' \text{ small.}$$

The identity of $\infty$-$Cat$ determines the universal coCartesian fibration with small fibers $p_{univ} : \infty\text{-}Cat_\bullet \to \infty\text{-}Cat$.

Quick observations:

1. In practice, we have a nice 1-category of (small) $\infty$-categories, but defining an $\infty$-category of (small) $\infty$-categories is something else.
2. Solving the first remark is somehow needed to make sense of straightening/unstraightening: if there is an equivalence, we should say in which sense. Assuming there is a $\infty$-category of (small) $\infty$-categories, is the straightening/unstraightening an equivalence of $\infty$-groupoids, or rather an equivalence of $\infty$-categories?

Straightening/unstraightening as an equivalence of $\infty$-groupoids really looks like solving a moduli problem: classifying coCartesian fibrations. As we will see later, it is also very strongly related to proving that there is a univalent universe of coCartesian fibrations.

All the problems mentioned above are known to have a solution explained in Lurie's book on higher topoi:

- One defines a Quillen equivalence between the Joyal model category structure and Bergner's model structure on the category of small simplicial categories.

$$\mathfrak{C}[-] : sSet \rightleftarrows sCat : N = \text{homotopy coherent nerve functor}$$

- One defines, for each quasi-category~~quasi-category~~ simplicial set $A$, a model structure on marked simplicial sets over $A$ whose fibrant objects are the coCartesian fibrations.
- One produces a simplicial Quillen equivalence between the coCartesian model structure on marked simplicial sets over $A$ and the projective model structure of simplicial functors from $\mathfrak{C}[A]$ to the simplicial category of small simplicial categories.

In this scenario, the homotopy coherent nerve functor plays central role. Riehl and Verity's approach to synthetic $\infty$-category theory heavily relies on homotopy coherent nerves as well, at least when it comes to (un)straightening.

We propose here an alternative approach: using the methods introduced by Voevodsky to produce a universe of Kan fibrations, we may introduce the universal coCartesian fibration with small fibers and build the theory of $\infty$-categories out of it. In this talk, we will consider quasi-categories as models of $\infty$-categories, but the aim is to be as model free as possible, by using the language of dependent type theory, or rather a fragment of it.

Indeed, dependent type theory is a suitable language to describe the theory of $\infty$-categories in the following sense:

- $\infty$-categories (quasi-categories) do form a Cartesian closed tribe in Joyal's sense;
- we do not have all dependent products (some functors are not Conduché);
- many features of $\infty$-category theory rely on the fact that dependent products along certain functors exist:
    1. for (co)Cartesian fibrations (i.e. maps classified by suitable universes)
    2. for the inclusion $\Delta^0 \amalg \Delta^0 = \partial\Delta^1 \to \Delta^1$: this one of the ways to define the join operation (out of which one defines (co)slices);
- there is a complex hierarchy of univalent universes: (co)Cartesian fibration, left/right fibrations... Corresponding to a hierarchy of deduction rules.

Deducing all the fundamental features of $\infty$-category theory out of the existence of univalent universes of coCartesian fibrations is thus a way to axiomatize $\infty$-category theory through dependent type theory. Such fundamental topics must include:

- the theory of (pointwise) Kan extension (adjunctions, (co)limits...)
- homotopy theory: constructing and computing localizations via calculus of fractions, derived functors (defined as Kan extensions)
- presentable $\infty$-categories and higher topoi.

The above should be the table of contents of a book entitled 'Directed Homotopy Type Theory: Univalent Foundations of Mathematics'.

Philosophical remark: this fits with Lawvere's point of view on the theory of categories: there is always a syntactic 1-category of types. We may see straightening/unstraightening as a way to relate the syntax (whatever may be expressed homotopy theoretically through the 1-category of (directed) types) and the properties of directed type of types (the universe).

In other words, we want directed homotopy type theory to speak of itself.

# Marked simplicial sets

### Definition

The category of marked simplicial sets $sSet^+$ is the category of pairs $(X, E_X)$ consisting of a simplicial set $X$ together with a subset of edges $E_X \subset X_1$ containing all identites (degeneracies).

- Given a simplicial set $A$, there is a marked simplicial set $A^\sharp = (A, \{\text{all edges}\})$. We simply write $sSet^+_{/A} = sSet^+_{/A^\sharp}$ for the slice over $A^\sharp$.
- Given a simplicial sets $C$, there is a marked simplicial set $C^\flat = (C, \{\text{all degeneracies}\})$.

# The (co)Cartesian model structure

### Theorem (Lurie)

*There is a unique model category structure on $sSet^{+}_{/A}$ whose cofibrations are the monomorphisms whose fibrant objects precisely are the coCartesian fibrations*

$$\begin{tikzcd} X \arrow[d, "p"'] \\ A \end{tikzcd}$$

*where the marked edges of $X$ are the coCartesian 1-morphisms.*

This is the coCartesian model structure. There is a dual version corresponding to Cartesian fibrations called the Cartesian model strucure. The assignment $X \mapsto X^{op}$ obviously induces a Quillen equivalence from the coCartesian model structure over $A$ to the Cartesian model structure over $A^{op}$.

# Marked left (right) anodyne extensions

**Definition**

We define $\mathcal{A}$ to be the smallest saturated class of maps $sSet^+$ containing the morphisms

(A1) $(\Lambda^n_k)^\flat \to (\Delta^n)^\flat$ for $n \geq 2$ and $0 < k < n$,

(A2) $J^\flat \to J^\sharp$ (with $J$ the 'walking isomorphism'),

(B1) $(\Delta^1)^\sharp \times (\Delta^1)^\flat \cup \{0\} \times (\Delta^1)^\sharp \to (\Delta^1)^\sharp \times (\Delta^1)^\sharp$,

(B2) $(\Delta^1)^\sharp \times (\partial\Delta^n)^\flat \cup \{0\} \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$ for $n \geq 0$.

The elements of $\mathcal{A}$ are called marked left anodyne extensions.

**Proposition (H. K. Nguyen)**

*Let $A$ be a simplicial set.*

- *Any marked left anodyne extension is a trivial cofibration of the coCartesian model structure over $A$.*
- *The right lifting property against marked left anodyne extensions determines fibrant objects as well as fibrations between fibrant objects in the coCartesian model structure over $A$.*

# Invariance properties of the (co)Cartesian model structure

The definition of marked left anodyne extensions above is equivalent to the one found in Lurie's HTT (3.1.1.1). As shown by Nguyen, the previous proposition is instrumental to give a homotopy-coherent-nerve-free proof of the basic functorial properties of the (co)Cartesian model structure.

**Theorem (Lurie)**

*Pulling back along any weak equivalence $f : A \to B$ of the Joyal model structure induces a right Quillen equivalence for the (co)Cartesian model structures.*

$$f^* : sSet^+_{/B} \to sSet^+_{/A}$$

The basic case is $i : \Lambda^n_k \to \Delta^n$ for $0 < k < n$.

# Cartesian fibrations are Conduché

Similarly, we get a homotopy-coherent-nerve-free proof of the fact that pulling back along a (co)Cartesian fibration gives homotopy pullbacks (and thus model free constructions).

**Theorem (Lurie, Higher Algebra, Appendix B.3)**

*Any Cartesian square of simplicial sets*

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ A' \arrow[r, "f"] & A \end{tikzcd}$$

*in which $p$ is a (co)Cartesian fibration is homotopy Cartesian in the Joyal model structure.*

# Fibers of (co)Cartesian fibrations

A marked version of Quillen's theorem A is:

**Theorem (H. K. Nguyen)**

*If $p : Y \to A$ is a Cartesian fibration, then the pullback functor*

$$p^* : sSet^+_{/A^\sharp} \to sSet^+_{/(Y,E_Y)} \text{ with } E_Y = \{coCartesian\ edges\}$$

*is a left Quillen functor on coCartesian model structures.*

**Corollary**

*Let $\mathbf{a}$ be an object in an $\infty$-category $A$. Then forming the fiber at $\mathbf{a}$ of coCartesian fibrations over $A$ is compatible with homotopy colimits.*

Indeed, up to a weak equivalence in $sSet^+$, this amounts to pulling back along the right (hence Cartesian) fibration $A_{/\mathbf{a}} \to A$ (and forgetting the edges is a right Quillen equivalence to the Joyal model structure).

# Universe of maps with small fibers 1

We fix a Grothendieck universe in order to speak of small sets. We define a simplicial set $\mathcal{U}$ whose $n$-simplices are the maps $p : X \to \Delta^n$ equipped, for each map $u : \Delta^m \to \Delta^n$, with a pullback square of the form

$$\begin{tikzcd} u^*X \arrow[r, "f_u"] \arrow[d, "u^*p"'] & X \arrow[d, "p"] \\ \Delta^m \arrow[r, "u"] & \Delta^n \end{tikzcd}$$

with $f_u = 1$ whenever $u = 1$, and such that each simplicial set $u^*X$ takes vakues in the given universe. There is a canonical map

$$p_{univ} : \mathcal{U}_\bullet \to \mathcal{U} \ , \quad (p, s) \mapsto p$$

where the $n$-simplices of $\mathcal{U}_\bullet$ are pairs $(p, s)$ with $p$ as above, and with $s$ a section of $p$.

# Universe of maps with small fibers 2

This means that, for a map $p : X \to Y$ with small fibers,

specifying pullback squares

$$\begin{tikzcd} Z \arrow[r, "t"] \arrow[d, "q"'] & X \arrow[d, "p"'] \\ \Delta^n \arrow[r, "s"] & Y \end{tikzcd}$$

for each simplex $s$ in $Y$, with $Z$ small

$\Leftrightarrow$

specifying a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & \mathcal{U}_\bullet \arrow[d, "p_{univ}"'] \\ Y \arrow[r, "F"] & \mathcal{U} \end{tikzcd}$$

# Universe of (co)Cartesian fibrations with small fibers

Since the property of being a coCartesian fibration (a left fibration=coCartesian fibration whose fibers are $\infty$-groupoids, Kan fibrations) is determined locally, we have a hierarchy of universes:

$$\begin{tikzcd}
k(\infty\text{-}Gpd_\bullet) \arrow[r, hook] \arrow[d, "k(p_{univ})"'] & \infty\text{-}Gpd_\bullet \arrow[r, hook] \arrow[d, "p_{univ}"'] & \infty\text{-}Cat_\bullet \arrow[r, hook] \arrow[d, "p_{univ}"'] & \mathfrak{U}_\bullet \arrow[d, "p_{univ}"'] \\
k(\infty\text{-}Gpd) \arrow[r, hook] & \infty\text{-}Gpd \arrow[r, hook] & \infty\text{-}Cat \arrow[r, hook] & \mathfrak{U}
\end{tikzcd}$$

with $k(C)$ being the maximal Kan complex of $C$. One can prove that $p_{univ} : \infty\text{-}Gpd_\bullet \to \infty\text{-}Gpd$ a univalent left fibration and that its codomain is indeed the $\infty$-category of small $\infty$-groupoids. One recovers from there that $k(p_{univ}) : k(\infty\text{-}Gpd_\bullet) \to k(\infty\text{-}Gpd)$ is Voevodsky's univalent universe of Kan fibrations.

# The $\infty$-category of small $\infty$-categories

## Theorem (H. K. Nguyen)

*The codomain $\infty$-Cat of the universal coCartesian fibration is an $\infty$-category.*

## Proof.

Using that inner horn inclusion induce Quillen equivalences of coCartesian model structures, we see that, for $0 < k < n$, for any coCartesian $p$ over $\Lambda^n_k$, we can find an homotopy pullback square with $q$ coCartesian:

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d, "p"'] & Y \arrow[d, dotted, "q"] \\ \Lambda^n_k \arrow[r, hook] & \Delta^n \end{tikzcd}$$

One rectifies this square into an actual pullback square using minimal fibrations. $\square$

# The universal (co)Cartesian functor 1

The Category $sSet^+_{/A}$ is canonically enriched over $sSet_{/A}$ and this makes the coCartesian model structure enriched in the sliced Joyal model structure.
Given two maps $p : X \to A$ and $q : Y \to A$ of marked simplicial sets, we denote by $\mathrm{Hom}_A(X, Y) \to A$ the corresponding Hom objects.
If $p$ and $q$ are coCartesian, this map is an isofibration and $\mathrm{Hom}_A(X, Y) = \mathrm{coCart}_A(X, Y)$ is the $\infty$-category of coCartesian functors $X \to Y$ over $A$. There is a simplicial subset $\mathrm{Equiv}_A(X, Y) \subset \mathrm{coCart}_A(X, Y)$ corresponding to coCartesian functors which are fiberwise equivalences of $\infty$-categories.

| a functor $B \to \mathrm{coCart}_A(X, Y)$ | $\Leftrightarrow$ | a map $B \to A$ together with a coCartesian functor over $B$ $B \times_A X \to B \times_A Y$ |
|---|---|---|

Both $\mathrm{coCart}_A(X, Y) \to A$ and $\mathrm{Equiv}_A(X, Y) \to A$ are isofibrations.

# The universal (co)Cartesian functor 2

We have a universal pair of coCartesian functors with small fibers:

$$\begin{tikzcd}
\infty\text{-}\mathit{Cat}^{(1)} \arrow[dr] & & \infty\text{-}\mathit{Cat}^{(2)} \arrow[dl] \\
& \infty\text{-}\mathit{Cat} \times \infty\text{-}\mathit{Cat} &
\end{tikzcd}$$

Hence a universal coCartesian functor

$$\begin{tikzcd}
\mathcal{X}^{(1)} \arrow[rr] \arrow[dr] & & \mathcal{X}^{(2)} \arrow[dl] \\
& \mathrm{coCart}_{\infty\text{-}\mathit{Cat}\times\infty\text{-}\mathit{Cat}}(\infty\text{-}\mathit{Cat}^{(1)}, \infty\text{-}\mathit{Cat}^{(2)}) &
\end{tikzcd}$$

where $\mathcal{X}^{(i)}$ is the pullback of $\infty\text{-}\mathit{Cat}^{(i)}$ for $i = 1, 2$.

# The universal (co)Cartesian functor 3

Finally, we get a universal equivalence of coCartesian fibrations with small fibers

$$\begin{tikzcd} \mathcal{Y}^{(1)} \arrow[rr, "\simeq"] \arrow[dr] & & \mathcal{Y}^{(2)} \arrow[dl] \\ & \mathrm{Equiv}_{\infty\text{-}\mathcal{C}at \times \infty\text{-}\mathcal{C}at}(\infty\text{-}\mathcal{C}at^{(1)}, \infty\text{-}\mathcal{C}at^{(2)}) & \end{tikzcd}$$

where $\mathcal{Y}^{(i)}$ is the restriction of $\mathcal{X}^{(i)}$ for $i = 1, 2$.

# Univalence

**Theorem (D.-C. C. & H. K. Nguyen)**

*The identity map* $\infty\text{-}\mathcal{C}at \to \mathrm{Equiv}_{\infty\text{-}\mathcal{C}at \times \infty\text{-}\mathcal{C}at}(\infty\text{-}\mathcal{C}at^{(1)}, \infty\text{-}\mathcal{C}at^{(2)})$ *is a trivial cofibration. Equivalently,* $\mathrm{Equiv}_{\infty\text{-}\mathcal{C}at \times \infty\text{-}\mathcal{C}at}(\infty\text{-}\mathcal{C}at^{(1)}, \infty\text{-}\mathcal{C}at^{(2)})$ *is a path object of* $\infty\text{-}\mathcal{C}at$ *in the Joyal model structure.*

The proof is similar to Voevodsky's for the univalence of the universal Kan fibration.

# Straightening/unstraightening 1

Let $A$ be a 1-category. There is a unique cocontinous simplicial functor

$$Fun(A, sSet) \to sSet^{+}_{/A}$$

defined on corepresentable functors through the assignment $\mathbf{a} \mapsto A_{\mathbf{a}/}$.

### Proposition

*The functor above is a left Quillen equivalence (from the injective model structure induced from Joyal's model structure to the coCartesian model structure).*

This is an easy exercise, once we observe that the total right derived functor of the right adjoint commutes with homotopy colimits, which follows from the analogous property for the operation of evaluating at the fibers of coCartesian fibrations.

### Corollary

*For any 1-category $B$, there is a Quillen equivalence from $Fun(B, Fun(A, sSet)) \cong Fun(A \times B, sSet)$ to $Fun(B, sSet^{+}_{/A})$.*

# Straightening/unstraightening 2

There is a 1-categorical universal coCartesian fibration $Cat_\bullet \to Cat$ whose restriction to $\Delta \subset Cat$ induces a coCartesian fibration $\Delta_\bullet \to \Delta$, which in turn is classified by a functor

$$i : \Delta \to \infty\text{-}Cat\,.$$

This induces a functor

$$\gamma : sSet \subset Fun(\Delta^{op}, \infty\text{-}Gpd) \xrightarrow{i_!} \infty\text{-}Cat\,.$$

The external version of straightening/unstraightening above together with univalence are instrumental to prove the following properties:

1. The $\infty$-category $\infty$-$Cat$ is cocomplete. This is why the functor $i_!$ above is well defined. Idem for its restriction $\gamma$.
2. The functor $\gamma$ sends the weak equivalences of the Joyal model structure to equivalences.
3. The functor $\gamma$ sends pushouts along monomorphisms to pushouts and commutes with sums.
4. The functor $\gamma$ induces an equivalence of the localization of the 1-category of $\infty$-categories by equivalences of $\infty$-categories to the 1-category $\tau(\infty\text{-}Cat)$.

# Straightening/unstraightening 3

This induces a new homotopy-coherent-nerve-free proof of the following theorem.

**Theorem (Lurie)**

*There is a canonical equivalence of $\infty$-categories*

$$L(Fun(A, sSet_{Joyal})) \cong Fun(A, L(sSet_{Joyal})) \cong Fun(A, \infty\text{-}Cat)$$

*for any 1-category $A$.*

Using that any $\infty$-category is the localization of a 1-category, it is easy to deduce

**Corollary (straightening/unstraightening)**

*For any simplicial set $A$, there is a canonical equivalence of $\infty$-categories:*

$$L(sSet^{+}_{/A}) \cong Fun(A, \infty\text{-}Cat)\,,$$

*where $L(sSet^{+}_{/A})$ is the localization of the coCartesian model structure.*

# Directed univalence

The universal co-Cartesian functors between coCartesian fibrations with small fibers thus uniquely determines a coCartesian fibration over $\Delta^1 \times \mathrm{coCart}_{\infty\text{-}Cat \times \infty\text{-}Cat}(\infty\text{-}Cat^{(1)}, \infty\text{-}Cat^{(2)})$ which is classified by a functor

$$\Delta^1 \times \mathrm{coCart}_{\infty\text{-}Cat \times \infty\text{-}Cat}(\infty\text{-}Cat^{(1)}, \infty\text{-}Cat^{(2)}) \to \infty\text{-}Cat.$$

This induces an equivalence of $\infty$-categories over $\infty\text{-}Cat \times \infty\text{-}Cat$:

$$\mathrm{coCart}_{\infty\text{-}Cat \times \infty\text{-}Cat}(\infty\text{-}Cat^{(1)}, \infty\text{-}Cat^{(2)}) \cong Fun(\Delta^1, \infty\text{-}Cat).$$

This is a type theoretic formulation of straightening/unstraightening and could be seen as a directed version of the property of univalence because restricting to equivalences in $\infty$-$Cat$ gives an equivalence over $\infty\text{-}Cat \times \infty\text{-}Cat$ which is just a reformulation of univalence for the universal coCartesian fibration:

$$\mathrm{Equiv}_{\infty\text{-}Cat \times \infty\text{-}Cat}(\infty\text{-}Cat^{(1)}, \infty\text{-}Cat^{(2)}) \cong \{\text{full subcategory of equivalences in } Fun(\Delta^1, \infty\text{-}Cat)\}.$$