Central H-spaces and banded types

$$(A \simeq A)_{(\mathrm{id})} \xrightarrow{\sim} A$$

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Our goals today

1^{st} **goal:** $\mathrm{HSpace}(A) \simeq (A \wedge A \to_* A)$ via **evaluation fibrations**

- generalises a classical formula of Arkowitz–Curjel [1] and Copeland [3] for spaces
- no H-space structures on $\mathbb{S}^{2n}$ for $n > 0$

2^{nd} **goal:** **centrality** and tensoring of **banded types**

- new construction of $\mathrm{K}(A, n)$ with H-space structure
- (applications to computation of Euler classes)

Most results formalised using the Coq-HoTT library [5].

Coherent H-spaces

Let A be a pointed type throughout, with $\mathrm{pt} : A$.

Def. A **(coherent) H-space structure** on A consists of:

- a binary operation $\mu : A \to A \to A$
- a left identity $\mu_l : \Pi_{a:A}\ \mu(\mathrm{pt}, a) = a$
- a right identity $\mu_r : \Pi_{a:A}\ \mu(a, \mathrm{pt}) = a$
- a **coherence** $\mu_{lr} : \mu_l(\mathrm{pt}) =_{\mu(\mathrm{pt},\mathrm{pt})=\mathrm{pt}} \mu_r(\mathrm{pt})$

We get a type HSpace($\boldsymbol{A}$) of (coherent) H-space structures on A.
(NB: The HoTT Book works with 'noncoherent' H-spaces!)

Eg. any group G, any ∞-group ΩX,
S^0, S^1, S^3, (S^7)

Evaluation fibrations

Def. Let $\alpha : B \to_* A$. The **evaluation fibration (of α)** is

$$\mathrm{ev}_\alpha(f, h) :\equiv f(\mathrm{pt}) : (B \to A)_{(\alpha)} \to_* A.$$

$$\sum_{f : B \to A} \| f = \alpha \|$$

Let A be connected.
Lemma. $\mathrm{HSpace}(A) \simeq \{$ pointed sections of $\mathrm{ev}_{\mathrm{id}}$ $\}$.

$$\begin{tikzcd} (A \simeq A)_{(\mathrm{id})} \arrow[d, "\mathrm{ev}_{\mathrm{id}}"] \\ A \end{tikzcd}$$

Prop. Any $\mu : \mathrm{HSpace}(A)$ induces a trivialisation of $\mathrm{ev}_{\mathrm{id}}$.

$$\begin{tikzcd} (A \to A) \arrow[rr, dashed, "\tau_\mu"] \arrow[dr, bend right, "\mathrm{ev}"'] & & (A \to A) \times A \arrow[dl, bend left, "\mathrm{pr}_2"] \\ & A & \end{tikzcd}$$

$$f \longmapsto \left(f(-)/f(\mathrm{pt}),\ f(\mathrm{pt}) \right)$$

Now restrict to path comp. $\square$

The moduli type of H-space structures

Prop. Any $\mu : \mathsf{HSpace}(A)$ induces a trivialisation of $\mathsf{ev}_{\mathsf{id}}$.

Prop.[1] Let A be an H-space. Then $\mathsf{HSpace}(A) \simeq (A \wedge A \to_* A)$.

Proof.

$$\mathsf{HSpace}(A) \simeq \{\text{pointed sects of } ev_{id}\}$$

$$\simeq A \to_* (A \to_* A, id)$$

$$\simeq A \to_* (A \to_* A, cst) \simeq (A \wedge A \to_* A)$$

$$\begin{tikzcd} A \vee A \arrow[r] \arrow[d] & A \times A \arrow[d] \\ 1 \arrow[r] & A \wedge A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\square$

Thus, e.g., $\mathsf{HSpace}(\mathbb{S}^1) \simeq \Omega^2\mathbb{S}^1 \simeq *$ and $\mathsf{HSpace}(\mathbb{S}^3) \simeq \Omega^6\mathbb{S}^3$.

$$\pi_6 \mathbb{S}^3 = \mathbb{Z}/12$$

(The H-space structure on $\mathbb{S}^3$ is due to Ulrik and Egbert [2].)

[1] Formula on path components due to Arkowitz–Curjel [1] and Copeland [3] for spaces.

H-space structures on even spheres

Lemma. $\mathrm{HSpace}(A) \simeq \{$ pointed sections of $\mathrm{ev}_{\mathrm{id}}$ $\}$, A connected.

Lemma.[2] Let $n, m > 1$ and $\alpha : \mathbb{S}^m \to_* \mathbb{S}^n$. If the evaluation fibration $\mathrm{ev}_\alpha : (\mathbb{S}^m \to \mathbb{S}^n)_{(\alpha)} \to_* \mathbb{S}^n$ merely admits a section, then the Whitehead product $[\alpha, \iota_n] : \pi_{n+m-1}(\mathbb{S}^n)$ vanishes.

Proof. Suppose $s : \mathbb{S}^n \to (\mathbb{S}^m \to \mathbb{S}^n)_{(\alpha)}$ is a section.

$$
\begin{tikzcd}
& \mathbb{S}^m \vee \mathbb{S}^n \arrow[dr] \arrow[drr, bend left, "\alpha \vee \iota_n"] & & \\
\mathbb{S}^{m+n-1} \arrow[ur] \arrow[dr] & & \mathbb{S}^m \times \mathbb{S}^n \arrow[r, "s'"] & \mathbb{S}^n \\
& 1 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}
$$

$[\iota_n, \iota_n] = 2$

□

Prop. There are no H-space structures on $\mathbb{S}^{2n}$ for $n > 0$.

[2] This is one direction of Lemma 2.2 in [4].

Central types

Lemma. $\mathrm{HSpace}(A) \simeq \{$ pointed sections of $\mathrm{ev}_{\mathrm{id}}$ $\}$, A connected.

Def. A pointed type A is **central** if $\mathrm{ev}_{\mathrm{id}}$ is an equivalence.

It follows that A is connected, $\mathrm{HSpace}(A)$ is contractible, and A is a "coherently abelian" H-space.

A **central H-space** is an H-space whose underlying type is central. We give conditions for when a connected H-space is central, e.g.:

Prop. Let X be a connected H-space.

$$X \text{ is central} \iff X \to_* \Omega X \text{ is contractible.}$$

Examples: $K(G,n)$ (for G abelian), $\mathbb{R}P^\infty \times \mathbb{C}P^\infty$

But not $S^1 \times \mathbb{C}P^\infty$.

Banded types

Suppose A is central, i.e., $\mathrm{ev}_{\mathrm{id}} : (A \simeq A)_{(\mathrm{id})} \xrightarrow{\sim}_* A$.

Def. $\mathrm{BAut}_1(A) :\equiv \Sigma_{X:\mathcal{U}} \overbrace{\|X = A\|_0}^{\text{band}}$ is the type of ***A*-bands**.

Lemma. $\Omega\,\mathrm{BAut}_1(A) \simeq \sum_{p:A=A} \|p = \mathrm{refl}\| \simeq (A \simeq A)_{(id)} \xrightarrow{\sim} A$.

We have an inversion operation $\mathrm{inv}(a) :\equiv \mathrm{pt}\,/\,a : A \to A$.

For an A-band X_p, its **dual** is $\boldsymbol{X_p^*} :\equiv (X, X \overset{p}{=} A \overset{\mathrm{inv}}{=} A)$.

Prop. $\boldsymbol{X_p} \otimes \boldsymbol{Y_q} :\equiv (X_P^* =_{\mathrm{BAut}_1} Y_q)$ is banded by A.

$(X_P^* =_{\mathrm{BAut}_1} Y_q) \longrightarrow (x = y)$

$(x \xrightarrow{p} A \xrightarrow{\mathrm{inv}} A \xrightarrow{q^{-1}} y)$

Proof.

WTS $\|(X_p^* =_{\mathrm{BAut}_1} Y_q) = A\|_0$, induct on p, q

$$(A = A)_{(\mathrm{inv})} \xrightarrow{\mathrm{inv}^*} (A \simeq A)_{(\mathrm{id})} \xrightarrow{\sim} A$$

□

The H-space structure on $\mathrm{BAut}_1(A)$

Suppose A is central, i.e., $\mathrm{ev}_{\mathrm{id}} : (A \simeq A)_{(\mathrm{id})} \xrightarrow{\sim}_* A$.

Prop. $X_p \otimes A_1 = X_p$ and $A_1 \otimes X_p = X_p$.
Proof.

$\square$

Theorem. $\mathrm{BAut}_1(A)$ is an abelian H-space for $\otimes$.

It's easy to show that $\mathrm{K}(G, n)$ is central (for G abelian), *for a given H-space* $\mathrm{K}(G, n)$. But we can also use this theorem to *construct* $\mathrm{K}(G, n)$, given some H-space $\mathrm{K}(G, 1)$.

Construction of $\mathrm{K}(G, n)$

Theorem. $\mathrm{BAut}_1(A)$ is an abelian H-space for $\otimes$.

Given a $\mathrm{K}(G, 1)$ with an H-space structure, inductively for $n \geq 1$:

1st, $K(G,n)$ is central

so $K(G,n+1) :\equiv \mathrm{BAut}_1(K(G,n))$

is an H-space, in fact central

since $K(G,n+1) \xrightarrow[*]{} K(G,n)$

is contractible.

Thank you for your attention!

References:

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