Homotopy canonicity for cubical type theory

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HoTT Electronic Seminar Talks
February 21st, 2019

[1] *Acknowledging support by the Centre for Advanced Studies Oslo*

Overview

1. Motivation:
   - Quick recap: cubical type theory
   - Canonicity
   - Choices in algorithms?
2. Cubical categories with families
   - Term model
   - Standard model
3. Sconing for cubical cwfs
4. Homotopy canonicity

Cubical type theory

1. Type theory based on a **constructive** model of HoTT using Kan cubical sets (with diagonals and connections).[2]
2. Currently very active area of research, several variants proposed[3]
3. Proof assistants:[4]
   `cubicaltt`, `Agda --cubical`, redtt, RedPRL, `yacctt`

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[2] Cohen, Coquand, H., Mörtberg at TYPES2015 – [CCHM]

[3] [AHW],[AH1+2],[ABCFHL],[CHM],[CM],. . . ; see also Carlo Angiuli's HoTTEST talk.

[4] See also Favonia's HoTTEST talk.

Cubical type theory

1. Formal **interval** $\mathbb{I}$

$$\frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \qquad \frac{}{0 : \mathbb{I}} \qquad \frac{}{1 : \mathbb{I}} \qquad \cdots$$

2. $i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \vdash u : A \mathrel{\widehat{=}}$ "u is an n-cube in A"
3. partial elements described by $\mathbb{F}$ (face lattice)

$$\frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (r = 0) : \mathbb{F}} \qquad \cdots$$

4. $\Gamma, \varphi \vdash u : A \mathrel{\widehat{=}}$ "u is a partial n-cube in A"
   (with shape described by φ)
   E.g.: $i : \mathbb{I}, (i = 0) \vee (i = 1) \vdash u : A$ given by two points.

Path types

The role of the identity type is played by path types:

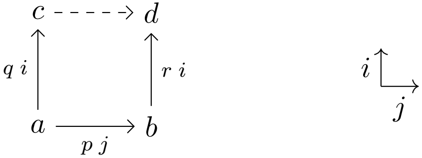
$$\frac{\Gamma, i : \mathbb{I} \vdash t(i) : A}{\Gamma \vdash \langle i \rangle\, t(i) : \mathsf{Path}_{i.A}\, t(0)\, t(1)} \qquad \cdots$$

Paths in A are like functions "$\mathbb{I} \to A$" with specified endpoints.

Directly justifies reflexivity and function extensionality.

Composition

The **composition operation** gives lids to “open boxes”. E.g.: The dashed line in



is the composition $\mathsf{comp}^i\ A\ [(j = 0) \mapsto q\ i, (j = 1) \mapsto r\ i]\ (p\ j)$.
Here $[(j = 0) \mapsto q\ i, (j = 1) \mapsto r\ i]$ describes a partial square with $\varphi = (j = 0) \vee (j = 1)$.

Special case of composition: for $i : \mathbb{I} \vdash A(i)$ we get $A(0) \to A(1)$.

Composition is explained by *induction* on the types!

Glue types

Glue types justify composition for **universes** as well as **univalence**.

They are allow to extend a partial equivalence

$$\varphi \vdash w : \mathsf{Equiv}(T, A) \qquad\qquad \text{(where } \varphi \vdash T\text{)}$$

over a total type A to a total equivalence $\mathsf{Equiv}(G, A)$.

Canonicity

Theorem (SH 2016)

Given a derivation of $i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \vdash u : \mathtt{N}$ *there exists* $m \in \mathbb{N}$ *such that* $i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \vdash u = \mathtt{S}^m\mathtt{0} : \mathtt{N}$.

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Ingredients of the proof:

- Deterministic and typed reduction relation:

$$i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \vdash t \longrightarrow u : A \qquad i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \vdash A \longrightarrow B$$

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- Computability predicates by **induction-recursion**:

$$i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \Vdash A$$
$$i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \Vdash u : A \quad \text{for } i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I} \Vdash A$$
$$\ldots$$

- Definition very involved since reduction not stable under intervall substitutions, e.g.:

$$I, i : \mathbb{I} \vdash t \longrightarrow u : A \not\Rightarrow I \vdash t[0/i] \longrightarrow u[0/i] : A[0/i]$$

- The "computational content" of univalence really lies in the algorithm for **compositions** and **glueing**.
- During developing cubical type theory we were faced to have various **choices** to take, especially when it comes to the explanation of composition for **glue**
- A priori these choices could lead to different results!
  Could we have $u : \mathtt{N}$ which with one choice computes to $0 : \mathtt{N}$ and in another choice to $17 : \mathtt{N}$ **?!**

Voevodsky's conjecture (2011)

Voevodsky conjectured homotopy canonicity for univalent type theory:

> *There is a terminating algorithm that for any $u : \mathbb{N}$ which is closed except that it may use the univalence axiom returns a closed numeral $n : \mathbb{N}$ not using the univalence axiom and a proof that $\mathsf{Id}_{\mathbb{N}}(u, n)$ (which may use the univalence axiom).*

Shulman [MSCS 2015] proves a truncated version; proof involves (Artin) glueing along the global section functor with the groupoid model (aka *sconing*).

Stumbling block for Voevodsky's conjecture: not clear how to glue with Kan simplicial set model as syntax doesn't directly give rise to a simplicial set

Cubical variation of Voevodsky's conjecture

Theorem

In the system **without** *computation rules for composition any* $u : \mathtt{N}$ *(closed) is* **path equal** *to a numeral* $\mathtt{S}^k\mathtt{0}$*.*

We prove this by a sconing construction.

Warning: this doesn't give a new proof of the corresponding statement for the system with computation rules.

Categories with families (cwf)

A **cwf** is given by:

- a category of *contexts* Con and substitutions $\mathsf{Hom}(\Delta, \Gamma)$;
- presheaves over the category of contexts:
  - Ty of *types*
- a presheaf Tm of *elements* over $\int_{\mathsf{Con}} \mathsf{Ty}$;
- a terminal object $\mathbf{1}$ in Con with $() \in \mathsf{Hom}(\Gamma, \mathbf{1})$;
- context extension $\Gamma.A$ for $\Gamma \in$ Con and $A \in \mathsf{Ty}(\Gamma)$ with $\mathsf{p} \in \mathsf{Hom}(\Gamma.A, \Gamma)$ and $\mathsf{q} \in \mathsf{Tm}(\Gamma.A, A\mathsf{p})$.
  Given $\sigma \in \mathsf{Hom}(\Delta, \Gamma)$ and $u \in \mathsf{Tm}(\Delta, A\sigma)$ we have $(\sigma, u) \in \mathsf{Hom}(\Delta, \Gamma.A)$.
  Moreover: $\mathsf{p}(\sigma, u) = \mathsf{p}$, $\mathsf{q}(\sigma, u) = u$, and $\sigma = (\mathsf{p}\sigma, \mathsf{q}\sigma)$

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- a category of *contexts* Con and substitutions $\mathsf{Hom}(\Delta, \Gamma)$;
- presheaves over the category of contexts:
  - Ty of *types*
  - $\mathsf{Ty}_n \subseteq \mathsf{Ty}$ (sub-presheaves), *types of level* $n \in \mathbb{N}$, which is cumulative $\mathsf{Ty}_n \subseteq \mathsf{Ty}_{n+1}$
- a presheaf Tm of *elements* over $\int_{\mathsf{Con}} \mathsf{Ty}$;
- a terminal object $\mathbf{1}$ in Con with $() \in \mathsf{Hom}(\Gamma, \mathbf{1})$;
- context extension $\Gamma.A$ for $\Gamma \in \mathsf{Con}$ and $A \in \mathsf{Ty}(\Gamma)$ with $\mathsf{p} \in \mathsf{Hom}(\Gamma.A, \Gamma)$ and $\mathsf{q} \in \mathsf{Tm}(\Gamma.A, A\mathsf{p})$.
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[5] Slightly modified for universes...

Categories with families: universes

For such a cwf to have universes means to have

- $\mathtt{U}_n \in \mathsf{Ty}_{n+1}(\Gamma)$ (stable under substitution) with
- $\mathsf{Ty}_n(\Gamma) = \mathsf{Tm}(\Gamma, \mathtt{U}_n)$ (such that the action of substitions is compatible)

(The structure of other type formers as usual, but is required to preserve universe levels.)

Internal language

Assume $\omega + 1$ Grothendieck universes and fix small category $\mathcal{C}$.

We will make use of the **internal language** of the presheaf topos like Orton-Pitts. This is a form of **extensional type theory**.[6]

Grothendieck universes lift to universes $\mathsf{U}_0, \mathsf{U}_1, \ldots, \mathsf{U}_\omega$ (cumulative) in presheaves over $\mathcal{C}$.

A propostion $\psi : \Omega$ gives rise to a subsingleton $\{* \mid \psi\}$ (also written as just ψ).

For an object X a **partial element** is given by $\psi : \Omega$ (its *extent*) and $u : \psi \to X$. Such a partial element extends $v : X$ if $\psi \Rightarrow u * = v$

[6] See also Andy Pitts' HoTTEST talk.

Internal language

Assumptions:

- an *interval* $\mathbb{I} : \mathsf{U}_0$ with endpoints $0, 1 : \mathbb{I}$
- an object $\mathbb{F} : \mathsf{U}_0$ of *cofibrant propositions* with a mono $[-] : \mathbb{F} \to \Omega$

For $A : \mathbb{I} \to \mathsf{U}_\omega$ define $\mathsf{hasFill}(A)$ as the type of operations:

$$\frac{\begin{array}{l}\varphi : \mathbb{F} \\ b \in \{0, 1\} \\ u : \Pi(i : \mathbb{I}).[\varphi] \vee (i = b) \to A\, i\end{array}}{s(\varphi, b, u) : \Pi(i : \mathbb{I}).A\, i \text{ extending } u \text{ on } \varphi}$$

The type of *filling structures* $\mathsf{Fill}(X, Y)$ for X and $Y : X \to \mathsf{U}_\omega$ is

$$\Pi(\gamma : \mathbb{I} \to X).\mathsf{hasFill}(Y \circ \gamma)$$

Cubical categories with families

A *cubical cwf* (w.r.t. $\mathcal{C}, \mathbb{I}, \mathbb{F}$) is given by:

- A cwf $(\mathsf{Con}, \mathsf{Hom}, \mathsf{Ty}, \mathsf{Tm}, \dots)$ **internally** to the presheaf topos on $\mathcal{C}$ equipped with Pi,Sigma, natural numbers, and universes, written $\mathtt{\Pi}$, $\mathtt{\Sigma}$, $\mathtt{N}$, $\mathtt{U}_n$
- a *filling operation*

  $$\mathtt{fill} : \mathsf{Fill}\big(\mathsf{Ty}(\Gamma), \lambda A.\mathsf{Tm}(\Gamma, A)\big)$$

  for Γ in Con (stable under substitutions $\sigma : \mathsf{Hom}(\Delta, \Gamma)$)
- dependent path types and glue types (next slides)

Cubical cwfs: (dependent) path types

We require operations:

$$\frac{A : \mathbb{I} \to \mathsf{Ty}(\Gamma) \qquad u_0 : \mathsf{Tm}(\Gamma, A\,0) \qquad u_1 : \mathsf{Tm}(\Gamma, A\,1)}{\mathtt{Path}(A, u_0, u_1) : \mathsf{Ty}(\Gamma)}$$

$$\frac{u : \Pi(i : \mathbb{I}).\mathsf{Tm}(\Gamma, A\,i)}{\mathtt{plam}(u) : \mathsf{Tm}(\Gamma, \mathtt{Path}(A, u\,0, u\,1))}$$

$$\frac{p : \mathsf{Tm}(\Gamma, \mathtt{Path}(A, u_0, u_1)) \qquad r : \mathbb{I}}{\mathtt{ap}(p, r) : \mathsf{Tm}(\Gamma, A\,r)} \qquad \begin{array}{l} \mathtt{ap}(p, 0) = u_0 \\ \mathtt{ap}(p, 1) = u_1 \end{array}$$

$$\mathtt{ap}(\mathtt{plam}(u), r) = u\,r \qquad \mathtt{plam}(\lambda i.\mathtt{ap}(p, i)) = p$$

(Plus: `Path` preserves levels, everything stable under substitution.)

Cubical cwfs: glue types

Using Path we can define equivalences via contractible fibers.

$$\frac{A : \mathsf{Ty}(\Gamma) \quad \varphi : \mathbb{F} \quad T : [\varphi] \to \mathsf{Ty}(\Gamma) \quad e : [\varphi] \to \mathsf{Tm}(\Gamma, \mathsf{Equiv}(T*, A))}{\mathtt{Glue}(A, \varphi, T, e) : \mathsf{Ty}(\Gamma) \text{ with } [\varphi] \to \mathtt{Glue}(A, \varphi, T, e) = T*}$$

$$\mathtt{unglue} : \mathsf{Tm}(\Gamma, \mathsf{Glue}(A, \varphi, T, e) \to A) \text{ with } [\varphi] \to \mathtt{unglue} = \mathtt{fst}(e)*$$

$$\frac{\begin{array}{l} a : \mathsf{Tm}(\Gamma, A) \\ t : [\varphi] \to \mathsf{Tm}(\Gamma, T) \\ [\varphi] \to \mathtt{app}(\mathtt{fst}(e)*, t*) = a \end{array}}{\begin{array}{c} \mathtt{glue}(a, t) : \mathsf{Tm}(\Gamma, \mathtt{Glue}(A, \varphi, T, e)) \\ \text{with } [\varphi] \to \mathtt{glue}(a, t) = t* \end{array}}$$

Satisfying some equations (β and η).

If we assume that $\mathbb{I}$ has connections and $\mathbb{F}$ forms sublattice of Ω_0 containing $(i = b)$ for $b \in \{0, 1\}$. Then, following [CCHM]:

- (non-dependent) `Path`-types gives rise to Martin-Löf identity types with propositional "computation rule" for `J`;
- a type is contractible whenever any partial element can be extended to a total one;
- one can show $\texttt{isEquiv}(\texttt{unglue})$ and with this univalence.

Thus: the underlying cwf models univalent type theory with propositional computation for `J`.

The term model

- Note: cubical cwf is almost a GAT (but subpresheaves)
- But we can concretely give an *initial* cubical cwf $\mathcal{T}$ (w.r.t. fixed $\mathcal{C}, \mathbb{I}, \mathbb{F}$ and strict morphisms)

The term model

- Note: cubical cwf is almost a GAT (but subpresheaves)
- But we can concretely give an *initial* cubical cwf $\mathcal{T}$ (w.r.t. fixed $\mathcal{C}, \mathbb{I}, \mathbb{F}$ and strict morphisms)
- $\mathcal{T}$ induced by judgments $\Gamma \vdash_X \mathcal{J}$ indexed by objects X in $\mathcal{C}$.

$$\frac{\Gamma \vdash_X \mathcal{J} \qquad f\colon Y \to X}{\Gamma f \vdash_Y \mathcal{J} f}$$

- In general, $\mathcal{T}$ will have infinitary rules like:

$$\frac{\begin{array}{l}\Gamma f \vdash_Y A_{f,r} \text{ for all } f\colon Y \to X, r \in \mathbb{I}(Y) \\ \Gamma f g \vdash_Z (A_{f,r})g = A_{fg,r} \text{ for all } f, g \\ \Gamma \vdash_X u_0 : A_{\mathsf{id},0} \qquad \Gamma \vdash_X u_1 : A_{\mathsf{id},1}\end{array}}{\Gamma \vdash_X \mathtt{Path}((A_{f,r}), u_0, u_1)}$$

- In setting of [CCHM]: possible to present rules in finitary way. ($\Gamma \vdash_{\{i,j,k\}} \mathcal{J}$ corresponds to $i : \mathbb{I}, j : \mathbb{I}, k : \mathbb{I}, \Gamma \vdash \mathcal{J}$ etc.)

The standard model

Assumptions:

1. **Internal**: $\mathbb{I}$ and $\mathbb{F}$ satisfy the axioms $\mathtt{ax}_1, \ldots, \mathtt{ax}_9$ of [OP].
2. **External**: "tiny interval", i.e. exponentiation with $\mathbb{I}$ has a right adjoint R (preserving levels)

Remark

- Item 2 is satisfied if $\mathbb{I}$ is representable and $\mathcal{C}$ closed under products (as in [CCHM]).
- $(-)^{\mathbb{I}} \dashv R$ cannot be made internal (cf. [LOPS]).

The standard model

1. Adjunction descends to an adjunction between categories of type over U_ω and $\mathsf{U}_\omega^{\mathbb{I}}$.[7]
2. Recall: global type $\mathsf{hasFill} : \mathsf{U}_\omega^{\mathbb{I}} \to \mathsf{U}_\omega$
3. Applying right adjoint to hasFill gives global $\mathsf{C} : \mathsf{U}_\omega \to \mathsf{U}_\omega$ s.t. for $X : \mathsf{U}_\omega$ and $Y : X \to \mathsf{U}_\omega$ global

   $$\begin{array}{c} \text{global sections of} \\ \mathsf{Fill}(X, Y) \end{array} \quad \leftrightarrow \quad \begin{array}{c} \text{global sections of} \\ \Pi(x : X).\mathsf{C}(Y\,x) \end{array}$$

   natural in X

[7] Details: see [LOPS] and Andy Pitts's HoTTEST talk

The standard model

- Using the internal language we can construct suitable operations for Path and Glue.
- Filling structures are closed under $\Pi, \Sigma, \mathsf{Path}, \mathsf{Glue}$.
- One can deduce that also C is closed under those type formers.
- Define $\mathsf{U}_n^{\mathsf{fib}}$ as $\Sigma(X : \mathsf{U}_n).\mathsf{C}(X)$.
- Using gluing we can show $\mathsf{C}(\mathsf{U}_n^{\mathsf{fib}})$.

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We obtain a cubical cwf $\mathcal{S}$ (called the standard model) where:

- The category of contexts is U_ω with $\mathsf{Hom}(\Delta, \Gamma)$ the functions Δ to Γ
- The types over Γ are maps $\langle A, \mathsf{fib}_A \rangle : \Gamma \to \mathsf{U}_\omega^{\mathsf{fib}}$.
- The elements of $\langle A, \mathsf{fib}_A \rangle$ are $\Pi(\rho : \Gamma).A\rho$.

Identity types in the standard model

Technique by Andrew Swan's gives **fibrant identity types**

$$\mathsf{Id}_A\, u_0\, u_1 \qquad \text{with} \qquad \mathsf{C}(\mathsf{Id}_A\, u_0\, u_1)$$

This is crucial for the treatment of natural numbers in the sconed model, as we can encode **fibrant indexed inductive types**!

Sconing

1. Same assumptions as for the standard model.
2. Let $\mathcal{M} = (\mathsf{Con}, \mathsf{Hom}, \dots)$ be a cubical cwf.
3. Goal: define new cubical cwf $\mathcal{M}^* = (\mathsf{Con}^*, \mathsf{Hom}^*, \dots)$ (the *sconing* of $\mathcal{M}$)

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5. Global sections operation $|-|$:
   - for $\Gamma : \mathsf{Con}$ define $|\Gamma| = \mathsf{Hom}(\mathbf{1}, \Gamma)$
   - for $\sigma : \mathsf{Hom}(\Delta, \Gamma)$ define $|\sigma| : |\Delta| \to |\Gamma|$ by $|\sigma|\rho = \sigma \circ \rho$
   - for $A : \mathsf{Ty}(\Gamma)$ define $|A| : |\Gamma| \to \mathsf{U}^{\mathsf{fib}}_{\omega}$ as

     $$|A| = \big(\mathsf{Tm}(\mathbf{1}, A()), k\,(A\rho)\big)$$

     Write also $|A|$ for $\mathsf{Tm}(\mathbf{1}, A())$.
   - for $u : \mathsf{Tm}(\Gamma, A)$ define $|u| : \Pi(\rho : \Gamma).|A|\rho$ as $|u|\rho = u\rho$

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   - for $u : \mathsf{Tm}(\Gamma, A)$ define $|u| : \Pi(\rho : \Gamma).|A|\rho$ as $|u|\rho = u\rho$
6. Could call $|-|$ weak cubical cwf morphism $\mathcal{M} \to \mathcal{S}$.

Sconing of $\mathcal{M}$

The cwf structure of $\mathcal{M}^*$:

- Contexts (Γ, Γ') : Con* given by Γ : Con and $\Gamma' : |\Gamma| \to \mathsf{U}_\omega$.
  Think of Γ' as **proof-relevant computability predicate**.
- Substitutions (σ, σ') : Hom$^*((\Delta, \Delta'), (\Gamma, \Gamma'))$ given by σ : Hom(Δ, Γ) and $\sigma' : \Pi(\rho : \Delta).\Delta'\,\rho \to \Gamma'(\sigma\rho)$.
- A type (A, A') : Ty$^*(\Gamma, \Gamma')$ consists of A : Ty(Γ) and

$$A' : \Pi(\rho : |\Gamma|)(\rho' : \Gamma'\rho).|A|\rho \to \mathsf{U}^{\mathsf{fib}}_\omega.$$

  Usually write A' as $\langle A', \mathsf{fib}_{A'}\rangle$.
- Elements (u, u') : Tm$((\Gamma, \Gamma'), (A, \langle A', \mathsf{fib}_A\rangle))$ are given by u : Tm(Γ, A) and

$$u' : \Pi(\rho : |\Gamma|)(\rho' : \Gamma'\,\rho).A'\,\rho\,\rho'\,(u\rho).$$

Context extensions in $\mathcal{M}^*$

$(\Gamma, \Gamma').(A, \langle A', \mathsf{fib}_{A'} \rangle)$ is defined as $(\Gamma.A, (\Gamma.A)')$ where

$$(\Gamma.A)'(\rho, a) = \Sigma(\rho' : \Gamma' \, \rho).A' \, \rho \, \rho' \, a$$

with

- first projection $\mathsf{p}^* = (\mathsf{p}, \mathsf{p}')$, $\mathsf{p}'(\rho, a)(\rho', a') = \rho'$, and
- second projection $\mathsf{q}^* = (\mathsf{q}, \mathsf{q}')$, $\mathsf{q}(\rho, a)(\rho', a') = a'$.

(The rest of the cwf can also be defined.)

Dependent products in $\mathcal{M}^*$

Let

- $(A, \langle A', \mathsf{fib}_{A'} \rangle) : \mathsf{Ty}^*(\Gamma, \Gamma')$, and
- $(B, \langle B', \mathsf{fib}_{B'} \rangle) : \mathsf{Ty}^*((\Gamma, \Gamma').(A, \langle A', \mathsf{fib}_{A'} \rangle))$.

Define the dependent product $\Pi^*((A, \langle A', \mathsf{fib}_{A'} \rangle), (B, \langle B', \mathsf{fib}_{B'} \rangle))$ by

$$(\Pi(A, B), \langle \Pi(A, B)', \mathsf{fib}_{\Pi(A,B)'} \rangle)$$

where $\Pi(A, B)' \, \rho \, \rho' \, f$ is

$$\Pi(a : |A| \, \rho)(a' : A' \, \rho \, \rho' \, a).B' \, (\rho, a) \, (\rho', a') \, (\mathtt{app}(f, a)).$$

This type is fibrant since C is closed under dependent products!

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This type is fibrant since C is closed under dependent products! Intro and elimination uses:

- $\mathtt{lam}(b)' \, \rho \, \rho' \, a \, a' = b' \, (\rho, a) \, (\rho', a')$
- $\mathtt{app}(f, a)' \, \rho, \rho' = f' \, \rho \, \rho' \, (a \, \rho) \, (a' \, \rho \, \rho')$

Other type formers

This also works similar for other type formers and operations.
Some important definitions:

1. universes $\mathtt{U}^*_n$: $\mathsf{Ty}^*(\Gamma, \Gamma')$ given by $(\mathtt{U}_n, \langle \mathtt{U}'_n, \mathsf{fib} \rangle)$ where:

$$\mathtt{U}'_n \, \rho \, \rho' \, A = |A| \rho \to \mathsf{U}^{\mathsf{fib}}_n$$

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Other type formers

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Crucial: we don't have to check any equations for how `fill` is defined for each type!

Natural numbers in $\mathcal{M}^*$

Natural numbers $\mathtt{N}^* = (\mathtt{N}^*, \mathtt{N}')$ in $\mathcal{M}^*$ are defined as an *indexed* inductive family over $|\mathtt{N}|$ with constructors:

- $0' : \mathtt{N}'\, \mathtt{0}$
- $\mathsf{S}' : \Pi(n : |\mathtt{N}|\, \rho).\mathtt{N}'\, n \to \mathtt{N}'\, (\mathtt{S}\, n)$

Can be encoded as a "parametrized" inductive type $\mathtt{N}'\, n$ where, e.g., the zero constructor has type:

$$0'' : \mathsf{Id}_{|\mathtt{N}|}\, n\, 0 \to \mathtt{N}'\, n$$

The use of Id is crucial for $\mathtt{N}'$ to be fibrant!

Simpler than treatment in Shulman's construction (which involves LEM for `natrec`).

Theorem

$\mathcal{M}^*$ is a cubical cwf and the first projection gives a morphism $\mathcal{M}^* \to \mathcal{M}$ of cubical cwfs.

Homotopy canonicity

Theorem

With the same assumptions on $\mathcal{C}, \mathbb{I}, \mathbb{F}$ as for the standard model: given a closed numeral $n : \mathsf{Tm}(\mathbf{1}, \mathtt{N})$ in the term model $\mathcal{T}$, we have a numeral $k : \mathbb{N}$ and a path $p : \mathsf{Tm}(1, \mathtt{Path}(\mathtt{N}, n, \mathtt{S}^k \mathtt{0}))$.

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- By induction n' : $\mathtt{N}'\, n$ gives $k : \mathbb{N}$ and a witness $\mathsf{Id}_{|\mathtt{N}|}\, n\, (\mathtt{S}^k \mathtt{0})$, and hence q : $\mathsf{Path}_{|\mathtt{N}|}\, n\, (\mathtt{S}^k \mathtt{0})$.

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- Build sconing $\mathcal{T}^*$ of the term model $\mathcal{T}$.
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- Set $p = \mathtt{plam}(\lambda i. q\, i)$. □

Extensions

- Sconing can be extended for cubical cwfs with identity types
- ... and for higher inductive types (of [CHM]).
- Also works if glue types are replaced by axioms $\mathsf{Tm}(\Gamma, \mathtt{iUnivalence}_n)$ for univalence.

Simplicial set model

Consider $\mathcal{C} = \Delta$ the simplex category, $\mathbb{I} = \Delta^1$ and $\mathbb{F}$ sublattice of Ω of decidable sieves.

The axioms needed for the standard model are **not** satisfied ($\mathbb{I}$ not tiny).

Assuming the *law of excluded middle*, we can still define $\mathsf{C} : \mathsf{U}_\omega \to \mathsf{U}_\omega$ s.t. we have maps[8]

$$\text{global sections of } \mathsf{Fill}(X, Y) \quad \leftrightarrow \quad \text{global sections of } \Pi(x : X).\mathsf{C}(Y\,x)$$

which allows us to adapt the argument for $\mathcal{S}_{\mathsf{sSet}}$ to simplicial sets.

[8] For details see Appendix D of our preprint.

Simplicial set model

Kapulkin-Voevodsky: can see simplicial sets as full subtopos of distributive lattice cubical sets.

This gives rise to functor from cubical cwfs over simplicial sets to cubical cwfs over distributive lattice cubical sets.

Thus, $\mathcal{S}_{\mathsf{sSet}}$ induces a model of distributive lattice cubical type theory (without computation rules for fill!), making it homotopically sound in the sense that we can only derive statements which also hold in $\mathcal{S}_{\mathsf{sSet}}$.

Conclusion

- We showed homotopy canonicity for cubical type theory without computation rules for composition using a sconing argument.
- Having no computation rules makes this argument easier.
- Crucial: identity types à la Swan
- Assuming classical logic, also applies to a simplicial setting.

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Thank you!

Some references

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