Acyclic types and epimorphisms in HoTT

Tom de Jong[1]

j.w.w. Ulrik Buchholtz[1] and Egbert Rijke[2]

[1]University of Nottingham
[2]University of Ljubljana

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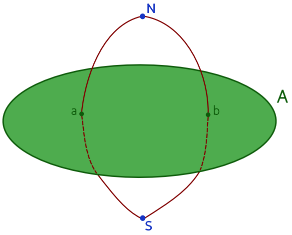
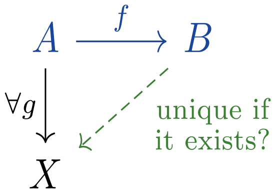




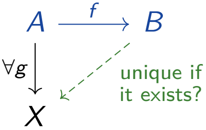
Epimorphisms

- In 1-category theory, a map $f : A \to B$ is an epi(morphism) if for every $g, h : B \to C$ we have

$$g \circ f = h \circ f \implies g = h.$$

  In other words, $(-) \circ f$ is an injection.

- Note:



$\iff$ f is an epi

- Def. A map $f : A \to B$ is an epi if $(-) \circ f$ is an embedding.

Epis w.r.t. k-types

- Def. (repeated) A map $f : A \to B$ is an epi if for every type X, the map

$$(B \to X) \xhookrightarrow{(-)\circ f} (A \to X)$$

is an embedding.

- Def. A map $f : A \to B$ is an epi w.r.t. k-types if for every k-type X, the map

$$(B \to X) \xhookrightarrow{(-)\circ f} (A \to X)$$

is an embedding.

- Lemma A map $f : A \to B$ is an epi w.r.t. k-types if and only if its k-truncation $\|f\|_k : \|A\|_k \to \|B\|_k$ is.

Basic properties of epis

- If $f : A \to B$ is an epi/k-epi, then the composite $A \xrightarrow{f} B \xrightarrow{g} C$ is an epi/k-epi if and only if g is.

- Every equivalence is an epi and every k-equivalence is a k-epi. Hence, every k-connected map is a k-epi.

- A map $f : A \to B$ is an epi if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array}$$

  is a pushout.

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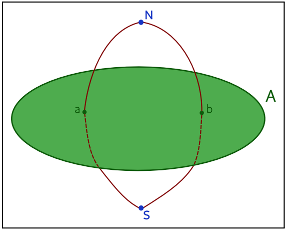
- A map $f : A \to B$ is a 0-epi if and only if it is a surjection.

- To see what happens for $k > 0$, we turn to acyclic types.

Acyclic types

- Def. The suspension ΣA of a type A is the pushout

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ \downarrow & \ulcorner & \downarrow N \\ \mathbf{1} & \underset{S}{\longrightarrow} & \Sigma A \end{array}$$



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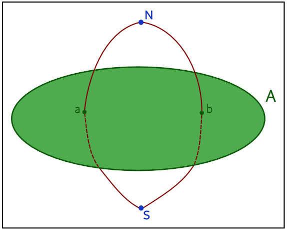


- Def. A type A is acyclic if ΣA is contractible, and k-acyclic if ΣA is k-connected (i.e. $\|\Sigma A\|_k$ is contractible).

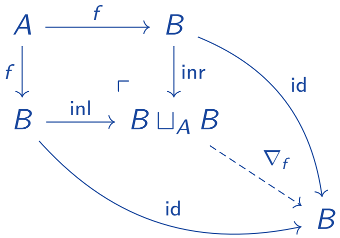
- Ex. A type is 0-acyclic if and only if it is inhabited.

- Ex. Every k-connected type is $(k+1)$-acyclic, so the circle $\mathbb{S}^1$ is 1-acyclic.

Acyclic maps

- Def. A map $f : A \to B$ is (k-)acylic if all of its fibres are.
  (Recall: $\mathrm{fib}_f(b) \equiv \sum_{a:A} f(a) = b$.)

- Lemma A map $f : A \to B$ is acylic/k-acylic if and only if its codiagonal ∇_f is an equivalence/k-connected.



Proof. For every $b : B$, we have $\Sigma\, \mathrm{fib}_f(b) \simeq \mathrm{fib}_{\nabla_f}(b)$. □

The epimorphisms are the acyclic maps

- Thm. A map is an (k-)epi if and only if it is (k-)acylic.

  Proof. $f : A \to B$ is an epi $\iff$ $\begin{array}{ccc} A & \xrightarrow{f} & B \\ f\downarrow & & \downarrow\text{id} \\ B & \underset{\text{id}}{\to} & B \end{array}$ is a pushout

  $\iff$ $\nabla_f : B \sqcup_A B \to B$ is an equivalence

  $\iff$ f is acyclic. □

Perfect groups and k-acyclic sets

- Def. A (set-based) group G is perfect if its abelianisation is trivial. E.g., the group A_5 of even permutations on a 5-element set is perfect.

- Thm. For a group G, its classifying type $\mathrm{B}G$ is 2-acyclic if and only if G is perfect.

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- Prop. A set is 1-acyclic if and only if it is contractible.

  Proof. Let G be the free group on a 1-acyclic set A with inclusion of generators $\eta : A \hookrightarrow G$. If A is 1-acyclic, then $A \to \mathbf{1}$ is a 1-epi, so the constant map

  $$\mathrm{B}G \to (A \to \mathrm{B}G)$$

  is an embedding. Hence, the constant map $G \to (A \to G)$ is an equivalence. Thus, η is constant. But it is also an embedding, so A must be a subsingleton. Finally, A is also inhabited, because it is 0-acyclic. □

Characterising 1-acylic and 2-acyclic types

- Thm. A type is 1-acyclic if and only if it is connected.

  Proof. We already know that k-connected types are $(k+1)$-acyclic, so every connected type is 1-acyclic. Conversely, if A is 1-acyclic, then the composite

  $$A \xrightarrow{|-|_0} \|A\|_0 \to \mathbf{1}$$

  is a 1-epi. Further, $|-|_0$ is connected and hence a 1-epi. Thus, $\|A\|_0 \to \mathbf{1}$ is a 1-epi and $\|A\|_0$ is 1-acyclic. But this means that the set $\|A\|_0$ is contractible by the previous proposition. Hence, A is connected. □

- Cor. Every k-acylic type is connected for $k \geq 1$.

- Thm. A type A is 2-acyclic if and only if connected and $\pi_1(A, a)$ is perfect for every $a : A$.

Acyclic types and the Freudenthal suspension theorem

- Thm. Every 1-connected acyclic type is ∞-connected.

  Proof. By the Freudenthal suspension theorem, the unit $\sigma : A \to \Omega\Sigma A$ of the loop-suspension adjunction is $2n$-connected whenever A is n-connected (for $n \geq 0$).

  If A is acyclic, then $\Sigma A \simeq \mathbf{1}$, so $\Omega\Sigma A \simeq \mathbf{1}$, so the connectedness of σ is that of A.

  Now if A is also 1-connected, then σ, and hence A, is in turn 2-connected, then 4-connected, etc., hence 2^n-connected for all n. □

- Thm. A 1-connected type is $(k+1)$-acyclic if and only if it is k-connected.

The Higman group: a nontrivial acyclic type

- The Higman group is defined as the group with 4 generators a, b, c, d and 4 relations

$$r_a : a = [d, a] \quad r_b : b = [a, b] \quad r_c : c = [b, c] \quad r_d : d = [c, d],$$

  where $[x, y] \equiv xyx^{-1}y^{-1}$ denotes the commutator.
- In HoTT we can describe its classifying type $\mathrm{B}H$ as a HIT:

$$\begin{aligned} \mathrm{pt} &: \mathrm{B}H \\ a, b, c, d &: \mathrm{pt} = \mathrm{pt} \\ r_a &: a = [d, a] \\ r_b &: a = [a, b] \\ r_c &: a = [b, c] \\ r_d &: a = [c, d] \end{aligned}$$

Contractibility of $\Sigma\,\mathrm{B}H$ (1)

- We describe $\Sigma\,\mathrm{B}H$ as a HIT and simplify its description step-by-step.

$$N, S : \Sigma\,\mathrm{B}H$$
$$m_{\mathrm{pt}} : N = S$$
$$\vdots$$

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$$N, S : \Sigma\,\mathrm{B}H$$
$$m_{\mathrm{pt}} : N = S$$
$$\vdots$$

Contr. at (N, m_{pt}) $\rightsquigarrow$

$$N : \Sigma\,\mathrm{B}H$$
$$m_a, m_b, m_c, m_d : \mathrm{refl}_N = \mathrm{refl}_N$$
$$m_{r_a} : m_a = [m_d, m_a]$$
$$m_{r_b} : m_b = [m_a, m_b]$$
$$m_{r_c} : m_c = [m_b, m_c]$$
$$m_{r_d} : m_d = [m_c, m_d]$$

Contractibility of $\Sigma\, \mathrm{B}H$ (2)

$$
\begin{aligned}
N &: \Sigma\, \mathrm{B}H \\
m_a &: \mathsf{refl}_N = \mathsf{refl}_N \\
m_b &: \mathsf{refl}_N = \mathsf{refl}_N \\
&\vdots \\
m_{r_a} &: m_a = [m_d, m_a] \\
m_{r_b} &: m_b = [m_c, m_b] \\
&\vdots
\end{aligned}
$$

Contractibility of $\Sigma\,\mathrm{B}H$ (2)

- The crux is that commutators are trivial in higher homotopy groups by the Eckmann-Hilton argument.

$$
\begin{array}{rl}
N & : \Sigma\,\mathrm{B}H \\
m_a & : \mathsf{refl}_N = \mathsf{refl}_N \\
m_b & : \mathsf{refl}_N = \mathsf{refl}_N \\
& \vdots \\
m_{r_a} & : m_a = [m_d, m_a] \\
m_{r_b} & : m_b = [m_c, m_b] \\
& \vdots
\end{array}
\quad \overset{\text{Eckmann–Hilton}}{\rightsquigarrow} \quad
\begin{array}{rl}
N & : \Sigma\,\mathrm{B}H \\
m_a & : \mathsf{refl}_N = \mathsf{refl}_N \\
m_b & : \mathsf{refl}_N = \mathsf{refl}_N \\
& \vdots \\
m_{r_a} & : m_a = \mathsf{refl}_{\mathsf{refl}_N} \\
m_{r_b} & : m_b = \mathsf{refl}_{\mathsf{refl}_N} \\
& \vdots
\end{array}
$$

Contractibility of $\Sigma\,\mathrm{B}H$ (3)

$$
\begin{aligned}
N &: \Sigma\,\mathrm{B}H \\
m_a &: \mathsf{refl}_N = \mathsf{refl}_N \\
m_b &: \mathsf{refl}_N = \mathsf{refl}_N \\
&\vdots \\
m_{r_a} &: m_a = \mathsf{refl}_{\mathsf{refl}_N} \\
m_{r_b} &: m_b = \mathsf{refl}_{\mathsf{refl}_N} \\
&\vdots
\end{aligned}
$$

Contr. at (m_a, m_{r_a}), (m_b, m_{r_b}), etc. $\rightsquigarrow$

$$N : \Sigma\,\mathrm{B}H$$

- So $\Sigma\,\mathrm{B}H$ is equivalent to a single point and hence contractible, i.e. $\mathrm{B}H$ is acyclic.

Nontriviality of the Higman group

- We can repeat the above argument for n generators and n relations, yielding an acyclic type for all n.
- For $n \leq 3$, the resulting groups turn out to be trivial.
- The Higman group ($\mathsf{n} = 4$) is in fact infinite but proving this seems to require a bit of group theory.

Summary

At higher types the notion of epimorphism

- becomes quite strong,
- coincides with the notion of an acyclic map, and
- is interesting from the p.o.v. of synthetic homotopy theory.

Future work

- Do the acyclic maps form an accessible modality? (Classically, they do.)
- Plus construction in HoTT
- Kan-Thurston theorem in HoTT: every ∞-group can be presented by a pair (G, P) of a group G and perfect normal subgroup $P \triangleleft G$
- In further developments, can we work around needing Whitehead's principle (every ∞-connected type is contractible) or the weaker principle that every 1-connected acyclic type is contractible?
- Use the theory of binate groups to prove acyclicity of some infinitely presented groups
- Applications (where surjectivity is not sufficient)

References I

[1] G. Baumslag, E. Dyer and A. Heller. "The topology of discrete groups". In: *J. Pure Appl. Algebra* 16.1 (1980), pp. 1–47. DOI: 10.1016/0022-4049(80)90040-7.

[2] A. J. Berrick. "Universal groups, binate groups and acyclicity". In: *Group Theory. Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, June 8–19, 1987.* Ed. by Kai N. Cheng and Yu K. Leong. De Gruyter Proceedings in Mathematics. 1989. DOI: 10.1515/9783110848397-019.

[3] A. J. Berrick and Carles Casacuberta. "A universal space for plus-constructions". In: *Topology* 38.3 (1999), pp. 467–477. DOI: 10.1016/S0040-9383(97)00073-6.

[4] Marc Bezem, Thierry Coquand, Peter Dybjer and Martín Escardó. "Free groups in HoTT/UF in Agda". https://www.cs.bham.ac.uk/~mhe/TypeTopology/FreeGroup.html. 2021.

[5] J. Daniel Christensen and Egbert Rijke. "Characterizations of modalities and lex modalities". In: *Journal of Pure and Applied Algebra* 226.3, 106848 (2022). DOI: 10.1016/j.jpaa.2021.106848.

[6] P. Hall. "Some constructions for locally finite groups". In: *J. London Math. Soc.* 34 (1959), pp. 305–319. DOI: 10.1112/jlms/s1-34.3.305.

[7] Pierre de la Harpe and Dusa McDuff. "Acyclic groups of automorphisms". In: *Comment. Math. Helv.* 58.1 (1983), pp. 48–71. DOI: 10.1007/BF02564624.

[8] Jean-Claude Hausmann and Dale Husemoller. "Acyclic maps". In: *L'enseignement Mathématique* 25.1–2 (1979), pp. 53–75. DOI: 10.5169/seals-50372.

[9] Graham Higman. "A finitely generated infinite simple group". In: *J. London Math. Soc.* 26 (1951), pp. 61–64. DOI: 10.1112/jlms/s1-26.1.61.

[10] Marc Hoyois. "On Quillen's plus construction". 2019. URL: https://hoyois.app.uni-regensburg.de/papers/acyclic.pdf.

[11] D. M. Kan and W. P. Thurston. "Every connected space has the homology of a $K(\pi, 1)$". In: *Topology* 15.3 (1976), pp. 253–258. DOI: 10.1016/0040-9383(76)90040-9.

[12] Ray Mines, Fred Richman and Wim Ruitenburg. *A Course in Constructive Algebra*. Universitext. Springer, 1988. DOI: 10.1007/978-1-4419-8640-5.

References II

[13] Brian A. Munson and Ismar Volić. *Cubical Homotopy Theory*. Vol. 25. New Mathematical Monographs. Cambridge University Press, 2015. DOI: 10.1017/CBO9781139343329.

[14] George Raptis. "Some characterizations of acyclic maps". In: *J. Homotopy Relat. Struct.* 14.3 (2019), pp. 773–785. DOI: 10.1007/s40062-019-00231-6.

[15] Egbert Rijke. "Introduction to Homotopy Type Theory". Book draft; version of 8 April. 2022. URL: https://raw.githubusercontent.com/martinescardo/HoTTEST-Summer-School/main/HoTT/hott-intro.pdf.

[16] Luis Scoccola. "Nilpotent types and fracture squares in homotopy type theory". In: *Math. Structures Comput. Sci.* 30.5 (2020), pp. 511–544. DOI: 10.1017/s0960129520000146.

[17] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.