Final Coalgebras of Analytic Functors, in HoTT

Philipp Joram Niccolò Veltri

2022-12-15

Tallinn University of Technology

Introduction

Motivation

- Revisit *Adámek's Theorem*: sufficient conditions for an endofunctor to admit a final coalgebra.
- In HoTT: Which functors satisfy these conditions?
- Focus on *analytic* functors
- In particular: multiset functor

Applications

Applications

Categorical semantics of *coinductive types*:

- endofunctor: “signature” for a coinductive type
- final coalgebra: gives *co-recursion* principle

Applications

Categorical semantics of *coinductive types*:

- endofunctor: “signature” for a coinductive type
- final coalgebra: gives *co-recursion* principle

Dynamics of *transition systems/automata*:

- endofunctor F: space of reachable states
- coalgebra: $c : S \to F\,S$ transition function
- terminal coalgebra: gives *bisimulation* (=co-equivalence relation)

Before we start...

Please ask questions...

- about notation
- which foundations are in use currently (HoTT/UF, ZFC, ...?)
- anything else

Background

Coalgebras of endofunctors

Given an endofunctor $F : \mathcal{C} \to \mathcal{C}$:

- *F-coalgebra*: an object S and a morphism $c : S \to FS$
- F-coalgebra *morphism* $c \xrightarrow{f} c'$: commutative square

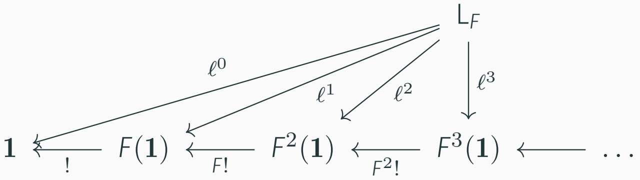
$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ \downarrow c & & \downarrow c' \\ FS & \xrightarrow{Ff} & FS' \end{array}$$

- $t : T \to FT$ is *final* iff $c \xrightarrow{\exists! f} t$ for all $c : S \to FS$

For any F, define the *final ω^{op}-chain*:

$$\mathbf{1} \xleftarrow{\;!\;} F(\mathbf{1}) \xleftarrow{\;F!\;} F^2(\mathbf{1}) \xleftarrow{\;F^2!\;} F^3(\mathbf{1}) \longleftarrow \ldots$$

L_F is the *limit* of this chain, i.e. a *final cone*:



$\mathsf{L}_F^{\mathsf{sh}}$: limit of *shifted chain*

$$F(\mathbf{1}) \xleftarrow[\;F!\;]{} F^2(\mathbf{1}) \xleftarrow[\;F^2!\;]{} F^3(\mathbf{1}) \xleftarrow[\;F^3!\;]{} F^4(\mathbf{1}) \longleftarrow \ldots$$

The Adámek-Pohlová theorem

Theorem (Pohlová '73, Adámek '74)

If L_F *exists and preserves this limit, then*

1. *there exists a coalgebra* $\mathsf{L}_F \to F(\mathsf{L}_F)$
2. *this coalgebra is final*

Proof (idea).

- F preserves $\mathsf{L}_F \Rightarrow \mathrm{pres}_F : F(\mathsf{L}_F) \cong \mathsf{L}_F^{\mathrm{sh}}$
- abstract nonsense $\Rightarrow \mathsf{L}_F \cong \mathsf{L}_F^{\mathrm{sh}}$
- the coalgebra:

$$\mathrm{fix} : \; \mathsf{L}_F \longrightarrow \mathsf{L}_F^{\mathrm{sh}} \xrightarrow{\mathrm{pres}_F^{-1}} F(\mathsf{L}_F)$$

□

Analytic functors

Generalization of *polynomial* functors ("sums-of-products"), due to Joyal [4].

Definition (Normal form, Hasegawa [3])

A set-endofunctor F is *analytic* if it is of the form

$$FX =_{\mathrm{df}} \sum_{a \in A} X^{B(a)} / G(a)$$

for a set A, family of finite sets $\{B(a)\}_{a \in A}$ and subgroups $G(a) \leq \mathrm{Aut}(B(a))$ where

$$v \sim_a w =_{\mathrm{df}} \exists \sigma \in G(a).\ (v_1, \ldots, v_k) = (w_{\sigma(1)}, \ldots, w_{\sigma(k)})$$

Summary

Encoding in HoTT:

- all definitions work for "functors" $F : \mathsf{Type} \to \mathsf{Type}$
- $\mathsf{L}_F : \mathsf{Type}$ and $\mathsf{pres}_F : F(\mathsf{L}_F) \to \mathsf{L}_F^{\mathsf{sh}}$ always exist
- no restrictions on h-level necessary (yet)

Our questions:

- Is pres_F an isomorphism, *constructively*?
- Is fix still *final*?

Finite multiset functors in hSet

Finite multisets

A multiset...

- is a collection of stuff of a sort X
- keeps track of multiplicity
- does not care about order of stuff

Example:

$$\{\text{pastry}, \text{coffee}\} = \{\text{coffee}, \text{pastry}\} \neq \{\text{coffee}, \text{pastry}, \text{coffee}\}$$

Finite multisets as HITs

Finite multisets = *free commutative monoid* over a type X

Finite multisets as HITs

Finite multisets = *free commutative monoid* over a type X

Define a *higher inductive type* $\mathrm{FCM}\,X$ such that:

Finite multisets as HITs

Finite multisets = *free commutative monoid* over a type X

Define a *higher inductive type* $\mathsf{FCM}\,X$ such that:

- there's an empty multiset: $\emptyset : \mathsf{FCM}\,X$
- singletons are multisets: $\{\_\} : X \to \mathsf{FCM}\,X$
- binary unions exist: $\_ \cup \_ : \mathsf{FCM}\,X \to \mathsf{FCM}\,X \to \mathsf{FCM}\,X$

Finite multisets as HITs

Finite multisets = *free commutative monoid* over a type X

Define a *higher inductive type* $\mathsf{FCM}\,X$ such that:

- there's an empty multiset: $\emptyset : \mathsf{FCM}\,X$
- singletons are multisets: $\{\_\} : X \to \mathsf{FCM}\,X$
- binary unions exist: $\_ \cup \_ : \mathsf{FCM}\,X \to \mathsf{FCM}\,X \to \mathsf{FCM}\,X$

with path-constructors:

- $\emptyset$ is neutral wrt. $(\cup)$ and $(\cup)$ is commutative
- $\mathsf{FCM}\,X$ is an hSet

Finite multisets, alternatives

Other presentations of finite multisets include:

Finite multisets, alternatives

Other presentations of finite multisets include:

1. Lists set-quotiented by permutation

Finite multisets, alternatives

Other presentations of finite multisets include:

1. Lists set-quotiented by permutation
2. List-HIT with one path $x :: y :: xs = y :: x :: xs$

Finite multisets, alternatives

Other presentations of finite multisets include:

1. Lists set-quotiented by permutation
2. List-HIT with one path $x :: y :: xs = y :: x :: xs$
3. As an analytic functor (in hSet):

$$\mathsf{FMSet}\, X =_{\mathrm{df}} \sum_{k:\mathbb{N}} (\mathsf{Fin}\, k \to X) /_2 \sim_k$$

$$v \sim_k w =_{\mathrm{df}} \exists (\sigma : \mathsf{Fin}\, k \simeq \mathsf{Fin}\, k).\ v = w \circ \sigma$$

Finite multisets, alternatives

Other presentations of finite multisets include:

1. Lists set-quotiented by permutation
2. List-HIT with one path $x :: y :: xs = y :: x :: xs$
3. As an analytic functor (in hSet):

$$\mathsf{FMSet}\,X =_{\mathrm{df}} \sum_{k:\mathbb{N}} (\mathsf{Fin}\,k \to X) \,/_2 \sim_k$$

$$v \sim_k w =_{\mathrm{df}} \exists(\sigma : \mathsf{Fin}\,k \simeq \mathsf{Fin}\,k).\ v = w \circ \sigma$$

Finite multisets, alternatives

Other presentations of finite multisets include:

1. Lists set-quotiented by permutation
2. List-HIT with one path $x :: y :: xs = y :: x :: xs$
3. As an analytic functor (in hSet):

$$\mathrm{FMSet}\,X =_{\mathrm{df}} \sum_{k:\mathbb{N}} (\mathrm{Fin}\,k \to X) /_2 \sim_k$$

$$v \sim_k w =_{\mathrm{df}} \exists(\sigma : \mathrm{Fin}\,k \simeq \mathrm{Fin}\,k).\ v = w \circ \sigma$$

Note

All presentations are (naturally) equivalent.

Prove theorems with convenient presentation, then transport.

Failure to apply Adámek's Theorem i

Theorem

$\mathsf{pres}_{\mathsf{FCM}} : \mathsf{FCM}(\mathsf{L}_{\mathsf{FCM}}) \to \mathsf{L}^{\mathsf{sh}}_{\mathsf{FCM}}$ *is surjective, but injectivity is equivalent to LLPO.*

LLPO is a classical principle:

Definition (Lesser Limited Principle of Omniscience)

For any stream of booleans $b : \mathbb{N} \to \mathbb{B}$, if b is true at most once, then merely either

- all odd positions of b are false
- all even positions of b are false

Failure to apply Adámek's Theorem ii

Theorem

$\mathsf{pres}_{\mathsf{FCM}} : \mathsf{FCM}(\mathsf{L}_{\mathsf{FCM}}) \to \mathsf{L}^{\mathsf{sh}}_{\mathsf{FCM}}$ *is surjective, but injectivity is equivalent to LLPO.*

Proof (idea).

Injectivity $\Rightarrow$ LLPO:

1. from injectivity, (merely) extract paths of two-element multisets
2. use those to decide the above problem

□

Finite multiset functors in hGpd

Goal

What went wrong:

- permutations are *proof-relevant*
- set-truncation forgets about them

Instead, we work in hGpd:

1. Define multiset-like functor that keeps permutations around as *data*
2. show that it preserves limits and admits a final coalgebra
3. Relate it to FMSet

The type of bags

Definition

Define the type family $\mathsf{Bag} : \mathsf{Type} \to \mathsf{Type}_1$,

$$\mathsf{Bag}\, X =_{\mathsf{df}} \sum (B : \mathsf{FinSet}).\ \langle B \rangle \to X$$

with projections $\mathsf{card} = \pi_1$ and $\mathsf{el} = \pi_2$.

$B : \mathsf{FinSet}$ is a Bishop-finite set, $\langle B \rangle$ its underlying type.

Intuition

The type $\mathsf{Bag}\,X$ has the correct path type:

Lemma

For all $xs, ys : \mathsf{Bag}\,X$*,*

$$(xs = ys) \simeq \sum(\sigma : \mathsf{card}(xs) \simeq \mathsf{card}(ys)).\ \mathsf{el}(xs) = \mathsf{el}(ys) \circ \sigma$$

It is an endofunctor of hGpd:

Lemma

If X *is a groupoid, then so is* $\mathsf{Bag}\,X$*.*

Relation to FMSet

Is this a faithful generalization of finite multisets?

Theorem

For any X : Type, $\|\text{Bag}\, X\|_2 \cong \text{FMSet}\, X$.

Proof (sketch).

- standard use of eliminators
- For $\|\text{Bag}\, X\|_2 \to \text{FMSet}\, X$: define *weakly-constant* function to go from a proposition to a set.

$\square$

Bag preserves ω^{op}-chain limits

Theorem (Ahrens, Capriotti, and Spadotti [1])

For any $A : \mathrm{Type}$ *and* $B : A \to \mathrm{Type}$, *the polynomial functor*

$$P_{A,B}\, X =_{\mathrm{df}} \sum_{a:A} B\, a \to X$$

preserves ω^{op}*-chains and admits a final coalgebra.*

Insight: Proof by definition.

Corollary

Bag *is a polynomial functor:* $\mathrm{Bag} = P_{\mathrm{FinSet}, \langle \_ \rangle}$.

Therefore, $\mathrm{L}_{\mathrm{Bag}}$ *is carrier of the final* Bag*-coalgebra.*

Can we get a final FMSet-coalgebra from the one of Bag?

A final coalgebra for FMSet, after all? i

Application of Adámek's theorem to FMSet failed.

Does not mean that obtaining a final coalgebra is impossible:

Hypothesis

The (final) coalgebra structure L_{Bag} descends through set-truncation:

There is a final FMSet-coalgebra $\|L_{Bag}\|_2 \to \mathrm{FMSet}\,\|L_{Bag}\|_2$.

A final coalgebra for FMSet, after all? ii

A reassuring result (cf. Lambek's lemma):

Theorem

$\|\mathsf{L}_{\mathsf{Bag}}\|_2$ *is a fixpoint of* FMSet.

Proof.

$$\begin{aligned} \mathsf{FMSet}\,\|\mathsf{L}_{\mathsf{Bag}}\|_2 &\cong \mathsf{FMSet}\,\mathsf{L}_{\mathsf{Bag}} \\ &\cong \|\mathsf{Bag}\,\mathsf{L}_{\mathsf{Bag}}\|_2 \\ &\cong \|\mathsf{L}_{\mathsf{Bag}}\|_2 \end{aligned}$$

□

A not-so-reassuring result:

Theorem

Assuming $\mathsf{AC}_{\mathsf{hSet},\mathsf{hProp}}$ *and* $\mathsf{AC}_{\mathsf{hGpd},\mathsf{hSet}}$ *[5, Ex. 7.8],* $\mathsf{L}_{\mathsf{Bag}}$ *induces a final* FMSet*-coalgebra.*

Conjecture

The assumption of choice is *necessary*.

Proof (idea).

Derive an equivalence $(X \to \|\mathsf{Bag}\, X\|_2) \simeq (\|X \to \mathsf{Bag}\, X\|_2)$ from the assumption of choice. □

Conclusion

From our case-study of *multiset-like functors*:

- analytic functors in sets:
    - seem to be ill-behaved constructively
    - no Adámek's theorem without classical principles
- polynomial functors in groupoids:
    - admit final coalgebras
    - capture the right notion of symmetry

Thank you for your attention!

Cubical Agda code for many claims:

https://github.com/phijor/agda-cubical-multiset

References

[1] Benedikt Ahrens, Paolo Capriotti, and Régis Spadotti. "Non-wellfounded trees in Homotopy Type Theory". en. In: *Proceedings of TLCA 2015*. Vol. 38. LIPIcs. Apr. 2015, pp. 17–30. DOI: 10.4230/LIPICS.TLCA.2015.17. arXiv: 1504.02949 [cs.LO].

[2] Eric Finster et al. "A Cartesian Bicategory of Polynomial Functors in Homotopy Type Theory". In: *EPTCS 351, 2021, pp. 67-83* (Dec. 2021). DOI: 10.4204/EPTCS.351.5. arXiv: 2112.14050 [cs.LO]. URL: https://github.com/smimram/fibred-polynomials.

[3] Ryu Hasegawa. "Two applications of analytic functors". In: *Theor. Comput. Sci.* 272.1-2 (2002), pp. 113–175. DOI: 10.1016/S0304-3975(00)00349-2.

[4] André Joyal. “Foncteurs analytiques et espèces de structures”. In: *Combinatoire énumérative*. Springer Berlin Heidelberg, 1986, pp. 126–159. DOI: 10.1007/bfb0072514.

[5] The Univalent Foundations Program. *Homotopy type theory: Univalent foundations of mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

“Functors” in HoTT

A type-former $F : \text{Type} \to \text{Type}$ is a functor (in this talk’s sense) if

1. it has a functorial action on maps:

$$\text{map}_F : (f : X \to Y) \to (FX \to FY)$$

2. map_F preserves identities and composition up to a path

Some consolation

Theorem

$\mathsf{pres}_{\mathsf{FMSet}} : \mathsf{FMSet}(\mathsf{L}_{\mathsf{FMSet}}) \to \mathsf{L}^{\mathsf{sh}}_{\mathsf{FMSet}}$ *is surjective.*

Proof (sketch).

1. $t : \mathsf{FMSet}^n\,\mathbf{1}$ is an *unlabeled, unordered tree* of depth n.
2. $\mathsf{FMSet}^n\,\mathbf{1}$ is *linearly ordered*: lexicographic order on branching factor.
3. Use this to *find preimages* in the $\mathsf{pres}_{\mathsf{FMSet}}$-fibers (i.e. terms of $\mathsf{FMSet}(\mathsf{L}_{\mathsf{FMSet}})$).

$\square$

Small bags

Definition

Define the large type family $\text{Tote} : \text{Type} \to \text{Type}_1$,

$$\text{Tote}\, X =_{\text{df}} \sum (B : \text{FinSet}).\ \langle B \rangle \to X$$

with projections $\text{card} = \pi_1$ and $\text{el} = \pi_2$.

B : FinSet is a Bishop-finite set, $\langle B \rangle$ its underlying type.

In practice: use *equivalent, small type family*

$$\text{Bag} : \text{Type} \to \text{Type}$$

by axiomatizing a small skeleton $\text{Bij} \hookrightarrow \text{FinSet}$ (Finster et al. [2]).