Univalent Category Theory

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2022-10-06

The very beginning

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- Identities, composites, left/right unit, associativity.

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A good place to start: what is a category? A category $\mathcal{C}$ is..

- A **type** of objects $\mathbf{Ob}(\mathcal{C})$;
- For each $x, y : \mathbf{Ob}(\mathcal{C})$, a **type** of morphisms $\mathbf{Hom}_{\mathcal{C}}(x, y)$.
- Identities, composites, left/right unit, associativity.

But what's a "collection"? One attempt: a **type**.

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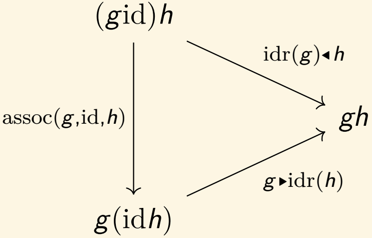

commutes.

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Solution(?): Simply pretend you don't see it?

Doing better

We call the data **Ob** + **Hom** (set-valued) + identities + composites + laws a **precategory**. Precategories don't care about the identity on objects (see no evil, speak no evil).

A **category** ("univalent category", "AKS-category") is a precategory $\mathcal{C}$ for which, for each $x : \mathbf{Ob}(\mathcal{C})$, the space of "isomorphs of x"

$$\sum_{y:\mathbf{Ob}(\mathcal{C})} x \cong y$$

is contractible.

Introduced in Ahrens *et al.*, 2013 as "saturated categories"; The HoTT book is behind just calling them "categories".

Why this makes sense

Requiring that the *space* of isomorphs of x be contractible makes sense categorically. Fix $\mathcal{C}$, $x : \mathbf{Ob}(\mathcal{C})$, and consider the full subcategory of $\mathcal{C}_{/x}$ on the objects $f : y \cong x$.

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This is a contractible groupoid! The terminal functor

$$! : \mathcal{C}_{/x}^{\cong} \to *$$

has a homotopy inverse

$$p : * \to \mathcal{C}_{/x}^{\cong}$$

which picks out the object (x, id). All other objects, by defn., are equipped with an iso to (x, id).

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- A fully faithful functor has subsingleton "essential fibres"; An essentially surjective functor has inhabited essential fibres. Between categories, essential fibres are just fibres, and eso+ff functors are just equivalences.

Why it's useful (2)

Univalence for categories is an instance of a more general framework of *identity systems*[1].

Slogan: An identity system is an implementation of J. Therefore, categories support *isomorphism induction*.

[1]See HoTT book §5.8; `1Lab.Path.IdentitySystem`

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Theorem (IsoJ)

*Fix a category $\mathcal{C}$, an object $x : \mathcal{C}$. For a type family $P : (y : \mathcal{C}) \to x \cong y \to$ **Type** to admit a section, it suffices to provide $p : P(x, \mathrm{id})$.*

Proof.

The space $\sum_{y:\mathcal{C}} x \cong y$ is contractible at (x, id), so you can transport p to $P(y, e)$ for any $e : x \cong y$. □

[1] See HoTT book §5.8; `1Lab.Path.IdentitySystem`

Displayed categories

More categorical structures can be made univalent!
Example: Displayed categories.

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- Identities over identities: $\mathrm{id}' : \mathbf{Hom}[\mathrm{id}](x, x)$;
- Composites over composites:
  $\circ' : \mathbf{Hom}[f](y', z') \to \mathbf{Hom}[g](x', y') \to \mathbf{Hom}[f \circ g](x', z')$.

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  $\circ' : \mathbf{Hom}[f](y', z') \to \mathbf{Hom}[g](x', y') \to \mathbf{Hom}[f \circ g](x', z')$.

Convention: x' lies over x.
Abbreviation: $\mathbf{Hom}[f](x', y')$ is clunky, we write $f' : x' \to_f y'$.

Displayed categories (2)

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Univalence for displayed categories; Over a univalent $\mathcal{B}$, t.f.a.e:

- Each fibre $\mathcal{E}^*(x)$ is a univalent category;
- For each $f : x \cong y$ and x', there is a contractible space of objects $\sum_{y':\mathbf{Ob}[y]}(x' \cong[f]\ y')$;
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If any of the above hold, we call $\mathcal{E} \mapsto \mathcal{B}$ a **displayed category**: it's an object in the slice $\mathbf{Cat}_{/B}$.

Displayed categories (3)

Theorem (Ahrens & Lumsdaine; 5.11)

If $\mathcal{E} \mapsto\!\!\triangleright\, \mathcal{B}$ is a displayed category, then it is an isofibration.

Note: Isofibrations can (and should) be thought of as "families of structures that respect isomorphism in the base".

Proof.

By `IsoJ`, to give Cartesian lifts for all $f : x \cong y$, it suffices to lift $\mathrm{id} : x \cong x$. But the identity map is Cartesian. □

Recent work & the future

- Ahrens *et. al*, 2019: Univalent and locally univalent bicategories; Displayed univalent bicategories(!)
- Ongoing work (in the 1Lab): (Displayed) univalent allegories
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Conjecture: Every "naturally-occurring variety of precategory" can be profitably split into "pre-" and "univalent" variations.

Thank you!