A higher encode decode method

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Overview

- Syllepsis
- Higher encode decode method
- A theorem about $K(\mathbb{Z}/2, n)$

Part 1. Syllepsis

Theorem (The Eckmann-Hilton argument)

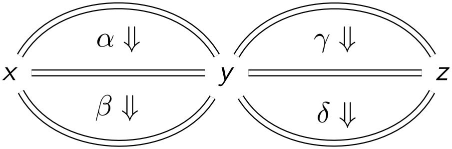
For any $k \geq 2$ and any pointed type X, the homotopy group

$$\pi_k(X)$$

is abelian.

Proof

There are two concatenation operations on paths of paths:



I.e., we have vertical composition $\alpha \bullet \beta$ and horizontal composition $\alpha \circ \gamma$. They satisfy

$$\text{interchange}(\alpha, \beta, \gamma, \delta) : ((\alpha \bullet \beta) \circ (\gamma \bullet \delta)) = ((\alpha \circ \gamma) \bullet (\beta \circ \delta))$$

Now we can construct an identification

$$\text{eckmann-hilton}(s, t) : s \bullet t = t \bullet s$$

for any $s, t : \text{refl} = \text{refl}$ by the following calculation:

$$\begin{aligned}
s \bullet t &= (s \circ \text{refl}) \bullet (\text{refl} \circ t) \\
&= (s \bullet \text{refl}) \circ (\text{refl} \bullet t) \\
&= s \circ t \\
&= (\text{refl} \bullet s) \circ (t \bullet \text{refl}) \\
&= (\text{refl} \circ t) \bullet (s \circ \text{refl}) \\
&= t \bullet s.
\end{aligned}$$

□

Syllepsis is an identification

$$\text{eckmann-hilton}(s,t) \bullet \text{eckmann-hilton}(t,s) = \text{refl}$$

In order to construct it, we will use the following:

- Three concatenation operations on the third identity type
- A fourth concatenation operation on the fourth identity type
- Unit laws for all of them
- Interchange laws between all of them
- Unit laws for the interchange law
- A coherence law between the three interchange laws
- Simplifications to the special case of Ω^3.
- Persistence

Kristina Sojakova recently formalized this result in a much more efficient way.

Definition

For any binary operation $f : A \to B \to C$ there is a binary action on paths

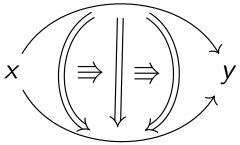
$$\text{ap-bin}_f : (x = x') \to (y = y') \to (f(x, y) = f(x', y')).$$

The binary action on paths induces n concatenation operations on the n-th identity type:

- For any $x, y, z : A$ we have

$$- \bullet - : (x = y) \to (y = z) \to (x = z).$$

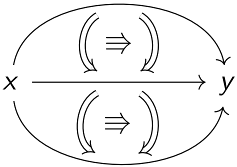
  On the third identity type, this gives a concatenation operation



- For any $x, y, z : A$ and any $p, p' : x = y$ and $q, q' : y = z$ we have

$$- \circ - : (p = p') \to (q = q') \to (p \bullet q = p' \circ q')$$

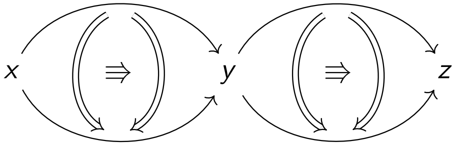
On the third identity type, this gives an operation



- On the third identity type, we can now define a third concatenation operation

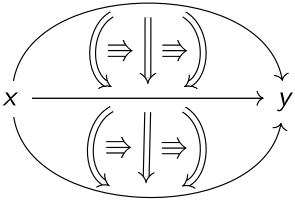
$$- * - : (\alpha = \alpha') \to (\beta = \beta') \to (\alpha \circ \beta = \alpha' \circ \beta')$$

  for any $\alpha, \alpha' : p = p'$, any $\beta, \beta' : q = q'$, $p, p' : x = y$ and $q, q' : y = z$.



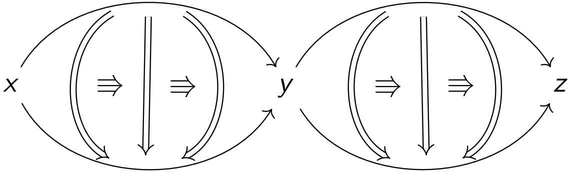
These definitions can be given uniformly by coinduction, using globular types.

We have three interchange laws, one for each pair of operations $\bullet$, $\circ$, and $*$:

- 

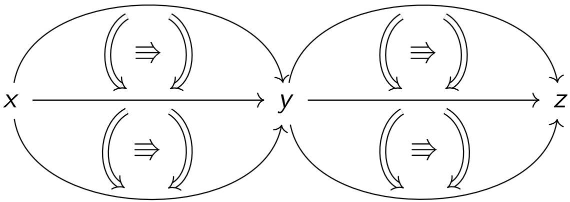
  For any $p, q, r : x = y$, any $\alpha, \beta, \gamma : p = q$, any $\delta, \epsilon, \zeta : q = r$, and any $\sigma : \alpha = \beta$, $\tau : \beta = \gamma$, $\nu : \delta = \epsilon$, and $\phi : \epsilon = \zeta$, we have an identification

$$(\sigma \bullet \tau) \circ (\nu \bullet \phi) = (\sigma \circ \nu) \bullet (\tau \circ \phi).$$

- 

For any $p, q : x = y$, $u, v : y = z$, $\alpha, \beta, \gamma : p = q$, $\delta, \epsilon, \zeta : u = v$, $\sigma : \alpha = \beta$, $\tau : \beta = \gamma$, $\nu : \delta = \epsilon$, and $\phi : \epsilon = \zeta$, we have an identification

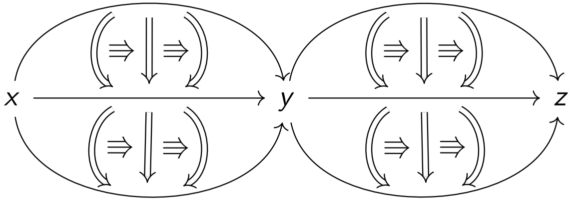
$$(\sigma \bullet \tau) * (\nu \bullet \phi) = (\sigma * \nu) \bullet (\tau * \phi).$$

- 

  and an interchange law that states that the square

$$\begin{array}{ccc}
(\alpha \bullet \gamma) \circ (\epsilon \bullet \eta) & \xlongequal{\text{interchange}(\alpha,\gamma,\epsilon,\eta)} & (\alpha \circ \epsilon) \bullet (\gamma \circ \eta) \\
{\scriptstyle (\sigma\circ\tau)*(\nu\circ\phi)} \Big\| & & \Big\| {\scriptstyle (\sigma*\nu)\circ(\tau*\phi)} \\
(\beta \bullet \delta) \circ (\zeta \bullet \theta) & \underset{\text{interchange}(\beta,\delta,\zeta,\theta)}{=\!=\!=\!=\!=} & (\beta \circ \zeta) \bullet (\delta \circ \theta)
\end{array}$$

  commutes.


x
y
z

Lemma

For any $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta : \Omega^3(X)$*, we have a commuting hexagon*

$$
\begin{array}{ccc}
 & ((\alpha \bullet \beta) \circ (\gamma \bullet \delta)) * ((\epsilon \bullet \zeta) \circ (\eta \bullet \theta)) & \\
\swarrow & & \searrow \\
((\alpha \bullet \beta) * (\epsilon \bullet \zeta)) \circ ((\gamma \bullet \delta) * (\eta \bullet \theta)) & & ((\alpha \circ \gamma) \bullet (\beta \circ \delta)) * ((\epsilon \circ \eta) \bullet (\zeta \circ \theta)) \\
\downarrow & & \downarrow \\
((\alpha * \epsilon) \bullet (\beta * \zeta)) \circ ((\gamma * \eta) \bullet (\delta * \theta)) & & ((\alpha \circ \gamma) * (\epsilon \circ \eta)) \bullet ((\beta \circ \delta) * (\zeta \circ \theta)) \\
\searrow & & \swarrow \\
 & ((\alpha * \epsilon) \circ (\gamma * \eta)) \bullet ((\beta * \zeta) \circ (\delta * \theta)) &
\end{array}
$$

Lemma

For any $s, t : \Omega^3(X)$, we have four commuting triangles:

$$
\begin{array}{ccc}
 & s * t & \\
\swarrow & & \searrow \\
s \circ t & \longrightarrow & s \bullet t
\end{array}
\qquad
\begin{array}{ccc}
 & s * t & \\
\swarrow & & \searrow \\
s \circ t & \longrightarrow & t \bullet s
\end{array}
$$

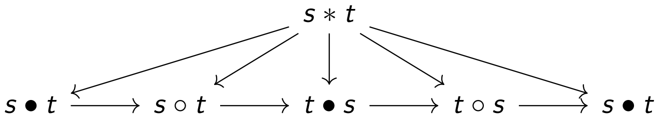
$$
\begin{array}{ccc}
 & s * t & \\
\swarrow & & \searrow \\
t \circ s & \longrightarrow & s \bullet t
\end{array}
\qquad
\begin{array}{ccc}
 & s * t & \\
\swarrow & & \searrow \\
t \circ s & \longrightarrow & t \bullet s
\end{array}
$$

Theorem (Syllepsis)

For any $s, t : \Omega^3(X)$, *we have*

$$\text{eckmann-hilton}(s, t) \bullet \text{eckmann-hilton}(t, s) = \text{refl}.$$

Proof.



□

Part 2. A higher encode decode method

The encode decode method

Theorem (The fundamental theorem of identity types)

Consider a type A with base point $a : A$. Let B be a type family over A equipped with a point $b : B(a)$. Then the following are equivalent:

1. *Any family of maps (in particular the canonical family of maps)*
$$(a = x) \to B(x)$$
indexed by $x : A$, is a family of equivalences.
2. *The type*
$$\sum_{(x:A)} B(x)$$
is contractible.
3. *The family B is a (unary) identity system on A.*

Theorem

Let X be a pointed type. Let P be a family of $(n+1)$-truncated types over $\|X\|_n$ equipped with a commuting triangle

$$\begin{array}{ccc} & & \sum_{(x:\|X\|_n)} P(x) \\ & \overset{f}{\nearrow} & \downarrow \\ X & \xrightarrow[\eta]{} & \|X\|_n. \end{array}$$

If f is $(n+1)$-connected, then

$$P(\eta(x_0)) \simeq K(\pi_{n+1}(X), n+1).$$

Proof.

The type $\sum_{(x:\|X\|_n)} P(x)$ is $(n+1)$-truncated, so any $(n+1)$-connected map into it is an $(n+1)$-truncation. Therefore we have

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \sum_{(x:\|X\|_n)} P(x) & & \xrightarrow{\simeq} & & \|X\|_{n+1} \\ & \searrow & & \swarrow & \\ & & \|X\|_n & & \end{array}$$

and by the bottom triangle we obtain the fiberwise equivalence that induces

$$P(\eta(x_0)) \simeq K(\pi_{n+1}(X), n+1).$$

□

The higher encode decode method

To show that $\pi_{n+1}(X) = G$, we can proceed as follows:

1. Define a pointed map

$$P : \|X\|_n \to \sum_{(X:\mathcal{U})} \|K(G, n+1) \simeq X\|$$

2. Construct a commuting triangle

$$\begin{array}{ccc} & & \sum_{(x:\|X\|_n)} P(x) \\ & \overset{f}{\nearrow} & \downarrow \\ X & \xrightarrow[\eta]{} & \|X\|_n \end{array}$$

   such that f is $(n+1)$-connected.

To apply this method in general, we need:

- A universal property of $\eta : X \to \|X\|_n$ with respect to $(n+2)$-types. In general, the map

$$(\|X\|_n \to Y) \to (X \to Y)$$

  is 0-truncated, if Y is $(n+2)$-truncated.
- A dependent universal property of $\eta : X \to \|X\|_n$ with respect to $(n+1)$-types.
- A good handle on the type

$$\mathrm{EM}(G, n) :\equiv \sum_{(X:\mathcal{U})} \|K(G, n) \simeq X\|.$$

The first two would be generalisations of results of Kraus. The space $\mathrm{EM}(G, n)$ is studied by Scoccola.

Part 3. A theorem about $K(\mathbb{Z}/2, n)$.

Theorem

$$K(\mathbb{Z}/2, n+1) \simeq \sum_{(X:\mathcal{U})} \|K(\mathbb{Z}/2, n) \simeq X\|$$

Theorem

$$K(\mathbb{Z}/2, n+1) \simeq \sum_{(X:\mathcal{U})} \|K(\mathbb{Z}/2, n) \simeq X\|$$

Lemma (Buchholtz, van Doorn, Rijke)

Let $n \geq 1$. For any two groups G and H (required to be both abelian in case $n \geq 2$) there is an equivalence

$$\mathsf{Grp}(G, H) \simeq \sum_{(f:K(G,n)\to K(H,n))} f(*) = *.$$

Furthermore, there is an equivalence

$$(G \cong H) \simeq \sum_{(e:K(G,n)\simeq K(H,n))} e(*) = *.$$

Proof. It suffices to show that the type

$$\sum_{(X:\mathcal{U})} \sum_{(p:\|K(\mathbb{Z}/2,n)\simeq X\|)} X$$

is contractible. In the case $n = 0$ this is a theorem of Buchholtz and Rijke. We may therefore assume $n > 0$, and in particular that $K(\mathbb{Z}/2, n)$ is connected.

- Center of contraction: $(K(\mathbb{Z}/2, n), \eta(\mathrm{id}), *)$.

- Contraction: Let $X : \mathcal{U}$ such that $\|K(\mathbb{Z}/2, n) \simeq X\|$ and $x : X$. Now it suffices to show that

$$\sum_{(e:K(\mathbb{Z}/2,n)\simeq X)} e(*) = x$$

is contractible. Since that is a proposition, we may assume $e : K(\mathbb{Z}/2, n) \simeq X$, and that $e(*) = x$. Therefore it suffices to show that

$$\sum_{(e:K(\mathbb{Z}/2,n)\simeq K(\mathbb{Z}/2,n))} e(*) = *$$

is contractible. By the lemma, this type is equivalent to the type of group isomorphisms

$$\mathbb{Z}/2 \cong \mathbb{Z}/2. \qquad \square$$

Corollary

For any $n : \mathbb{N}$ *we have*

$$K(\mathbb{Z}/2, n) \simeq (K(\mathbb{Z}/2, n) \simeq K(\mathbb{Z}/2, n)).$$

Corollary

Any fiber sequence $K(\mathbb{Z}/2,4) \hookrightarrow E \twoheadrightarrow \|S^3\|_3$ *is equivalently described by a map*

$$K(\mathbb{Z},3) \to K(\mathbb{Z}/2,5)$$