Syllepsis in Homotopy Type Theory

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Introduction

In Homotopy Type Theory, the following two properties hold:

- *Eckmann-Hilton* (Favonia, Christensen, Shulman, et al.): any two 2-loops $p, q : 1 = 1$ based at reflexivity commute.
- *Syllepsis* (S., Rijke): for any two 3-loops $p, q : 1_1 = 1_1$ based at reflexivity on reflexivity, the Eckmann-Hilton proof that q and p commute is the inverse of the Eckmann-Hilton proof that p and q commute.

The dimensions cannot be lowered: Eckmann-Hilton does not hold for 1-loops (counterexample: non-commuting endofunctions) and syllepsis does not hold for 2-loops (counterexample due to Vicary).

Outline

- Introduction
- Preliminaries
- The Eckmann-Hilton Proof
- Properties of The Eckmann-Hilton Proof
- Syllepsis
- Proof of Syllepsis: The Square, The Triangles, and The Result
- Future Work

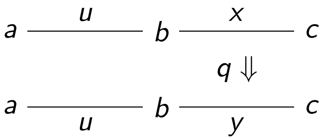
Whiskering

Lemma

For any points $a, b, c : A$, 1-paths $u : a = b$, $x, y : b = c$, and 2-path $q : x = y$, we have a term

$$\textit{whisk-L}(u, q) : u \cdot x = u \cdot y$$

Pictorially:



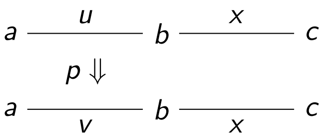
Whiskering

Lemma

For any points $a, b, c : A$*, 1-paths* $u, v : a = b$*,* $x : b = c$*, and 2-path* $p : u = v$*, we have a term*

$$\textit{whisk-R}(p, x) : u \cdot x = v \cdot x$$

Pictorially:



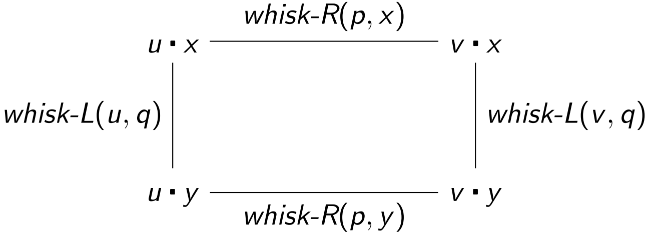
Whiskering Exchange Law

Lemma

For any points $a, b, c : A$*, 1-paths* $u, v : a = b$*,* $x, y : b = c$*, and 2-paths* $p : u = v$*,* $q : x = y$*, we have a term*

$$\textit{whisk-L-R}(p, q)$$

witnessing the commutativity of the diagram



Concatenation by Reflexivity is Natural

Lemma

Concatenation on the left by reflexivity is natural: for any points $a, b : A$, 1-paths $u, v : a = b$, and 2-path $p : u = v$, we have a term

$$\blacksquare\text{-}1\text{-}L\text{-}nat(p)$$

witnessing the commutativity of the diagram

$$\begin{array}{ccc}
1_a \cdot u & \overset{\blacksquare\text{-}1\text{-}L(u)}{\rule[0.5ex]{6em}{0.4pt}} & u \\
whisk\text{-}L(1_a, p) \Big| & & \Big| \; p \\
1_a \cdot v & \underset{\blacksquare\text{-}1\text{-}L(v)}{\rule[0.5ex]{6em}{0.4pt}} & v
\end{array}$$

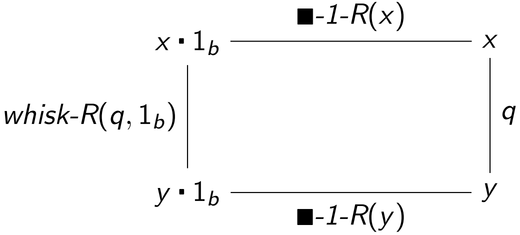
Concatenation by Reflexivity is Natural

Lemma

Concatenation on the right by reflexivity is natural: for any points $a, b : A$, 1-paths $x, y : a = b$, and 2-path $q : x = y$, we have a term

$$\blacksquare\text{-}\mathit{1}\text{-}R\text{-}nat(q)$$

witnessing the commutativity of the diagram



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The Eckmann-Hilton Proof

Theorem (Eckmann-Hilton)

For any point $\star : A$ and 2-loops $p, q : 1_\star = 1_\star$, we have a 3-path $EH(p, q)$:

$$p \cdot q$$

$$\Big|$$

$$\textit{whisk-L}(1_\star, p) \cdot \textit{whisk-R}(q, 1_\star)$$

$$\Big|$$

$$\textit{whisk-R}(q, 1_\star) \cdot \textit{whisk-L}(1_\star, p)$$

$$\Big|$$

$$q \cdot p$$

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Eckmann-Hilton on Reflexivity

The term $\mathrm{EH}(1, \mathrm{q})$ is equal to

$$1 \cdot q \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}L(q)} q \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}R(q)^{-1}} q \cdot 1$$

The term $\mathrm{EH}(\mathrm{p}, 1)$ is equal to

$$p \cdot 1 \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}R(p)} p \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}L(p)^{-1}} 1 \cdot p$$

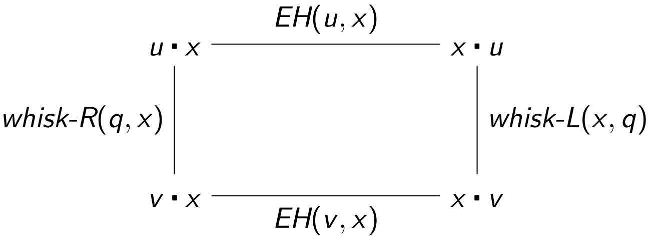
Naturality of Eckmann-Hilton

Lemma

For any 2-loops $u, v, x : 1 = 1$, and 3-path $q : u = v$, we have a term

$$EH\text{-}L\text{-}nat(q, x)$$

witnessing the commutativity of the diagram



Naturality of Eckmann-Hilton

Lemma

For any 2-loops $u, x, y : 1 = 1$, and 3-path $p : x = y$, we have a term

$$\textit{EH-R-nat}(u, p)$$

witnessing the commutativity of the diagram

$$\begin{array}{ccc}
u \cdot x & \overset{EH(u,x)}{=\!=\!=\!=\!=} & x \cdot u \\
{\scriptstyle \textit{whisk-L}(u,p)} \Big\| & & \Big\| {\scriptstyle \textit{whisk-R}(p,u)} \\
u \cdot y & \underset{EH(u,y)}{=\!=\!=\!=\!=} & y \cdot u
\end{array}$$

Naturality of Eckmann-Hilton Explicitly

The term $\mathrm{EH\text{-}L\text{-}nat}(q, 1_1)$ is equal to

$$
\begin{array}{ccccc}
1_1 \cdot 1_1 & \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}R(1_1)} & 1_1 & \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}L(1_1)^{-1}} & 1_1 \cdot 1_1 \\
{\scriptstyle \mathit{whisk}\text{-}R(q,1_1)} \Big| & & {\scriptstyle q} \Big| & & \Big| {\scriptstyle \mathit{whisk}\text{-}L(1_1,q)} \\
1_1 \cdot 1_1 & \xrightarrow[\blacksquare\text{-}\mathit{1}\text{-}R(1_1)]{} & 1_1 & \xrightarrow[\blacksquare\text{-}\mathit{1}\text{-}L(1_1)^{-1}]{} & 1_1 \cdot 1_1
\end{array}
$$

Naturality of Eckmann-Hilton Explicitly

The term $\text{EH-R-nat}(1_1, \text{p})$ is equal to

$$
\begin{array}{ccccc}
1_1 \cdot 1_1 & \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}L(1)} & 1_1 & \xrightarrow{\blacksquare\text{-}\mathit{1}\text{-}R(1)^{-1}} & 1_1 \cdot 1_1 \\
{\scriptstyle whisk\text{-}L(1,p)} \Big| & & {\scriptstyle p} \Big| & & \Big| {\scriptstyle whisk\text{-}R(p,1)} \\
1_1 \cdot 1_1 & \xrightarrow[\blacksquare\text{-}\mathit{1}\text{-}L(1)]{} & 1_1 & \xrightarrow[\blacksquare\text{-}\mathit{1}\text{-}R(1)^{-1}]{} & 1_1 \cdot 1_1
\end{array}
$$

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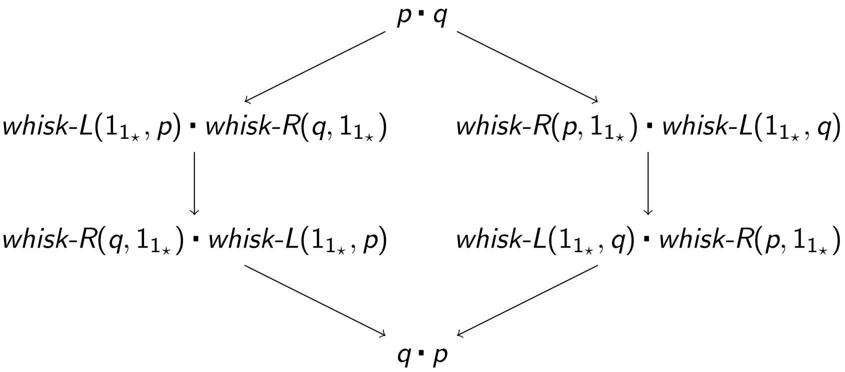
Syllepsis

Theorem

For any point $\star : A$ and 3-loops $p, q : 1_{1_\star} = 1_{1_\star}$, we have

$$EH(q, p) = EH(q, p)^{-1}$$



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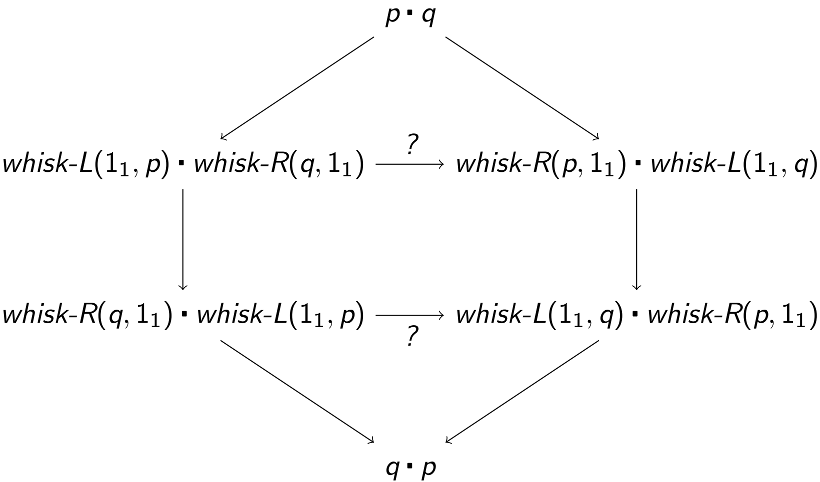
Syllepsis: The Square, The Triangles, and The Result

We can split the diagram as follows:



Syllepsis: The Square

Generalize to $p : x = y$ and $q : u = v$ for arbitrary 2-loops $x, y, u, v : 1 = 1$:

$$\begin{array}{ccc}
\mathit{whisk\text{-}L}(u, p) \cdot \mathit{whisk\text{-}R}(q, y) & \overset{?}{\text{———}} & \mathit{whisk\text{-}R}(p, u) \cdot \mathit{whisk\text{-}L}(y, q) \\
\Big| & & \Big| \\
\mathit{whisk\text{-}R}(q, x) \cdot \mathit{whisk\text{-}L}(v, p) & \underset{?}{\text{———}} & \mathit{whisk\text{-}L}(x, q) \cdot \mathit{whisk\text{-}R}(p, v)
\end{array}$$

Syllepsis: The Square

Generalize to $p : x = y$ and $q : u = v$ for arbitrary 2-loops $x, y, u, v : 1 = 1$:

$$\begin{array}{ccc}
\mathit{whisk\text{-}L}(u, p) \cdot \mathit{whisk\text{-}R}(q, y) & \overset{?}{\rule[0.5ex]{3em}{0.4pt}} & \mathit{whisk\text{-}R}(p, u) \cdot \mathit{whisk\text{-}L}(y, q) \\
\Big| & & \Big| \\
\mathit{whisk\text{-}R}(q, x) \cdot \mathit{whisk\text{-}L}(v, p) & \underset{?}{\rule[0.5ex]{3em}{0.4pt}} & \mathit{whisk\text{-}L}(x, q) \cdot \mathit{whisk\text{-}R}(p, v)
\end{array}$$

But: endpoints do not match!

Syllepsis: The Square

Generalize to $p : x = y$ and $q : u = v$ for arbitrary 2-loops $x, y, u, v : 1 = 1$:

$$
\begin{array}{ccc}
\mathit{whisk\text{-}L}(u, p) \cdot \mathit{whisk\text{-}R}(q, y) & \overset{?}{=\!=\!=} & \mathit{whisk\text{-}R}(p, u) \cdot \mathit{whisk\text{-}L}(y, q) \\
\Big| & & \Big| \\
\mathit{whisk\text{-}R}(q, x) \cdot \mathit{whisk\text{-}L}(v, p) & \underset{?}{=\!=\!=} & \mathit{whisk\text{-}L}(x, q) \cdot \mathit{whisk\text{-}R}(p, v)
\end{array}
$$

But: endpoints do not match! We need to insert Eckmann-Hilton.

Syllepsis: The Square

To construct the first horizontal path, we need to fill the following square:

$$
\begin{array}{ccc}
u \cdot x & \overset{EH(u,x)}{\rule{3cm}{0.4pt}} & x \cdot u \\
\mathit{whisk\text{-}L}(u,p) \Big| & & \Big| \mathit{whisk\text{-}R}(p,u) \\
u \cdot y & & y \cdot u \\
\mathit{whisk\text{-}R}(q,y) \Big| & & \Big| \mathit{whisk\text{-}L}(y,q) \\
v \cdot y & \underset{EH(v,y)}{\rule{3cm}{0.4pt}} & y \cdot v
\end{array}
$$

Syllepsis: The Square

We use the naturality of Eckmann-Hilton:

$$
\begin{array}{ccc}
u \cdot x & \overset{EH(u,x)}{\text{———}} & x \cdot u \\
\scriptstyle whisk\text{-}L(u,p) \Big| & & \Big| \scriptstyle whisk\text{-}R(p,u) \\
u \cdot y & \overset{EH(u,y)}{\text{———}} & y \cdot u \\
\scriptstyle whisk\text{-}R(q,y) \Big| & & \Big| \scriptstyle whisk\text{-}L(y,q) \\
v \cdot y & \underset{EH(v,y)}{\text{———}} & y \cdot v
\end{array}
$$

Syllepsis: The Square

To construct the second horizontal path, we need to fill the following square:

$$
\begin{array}{ccc}
u \cdot x & \overset{EH(u,x)}{\rule[0.5ex]{4em}{0.4pt}} & x \cdot u \\
\scriptstyle \textit{whisk-R}(q,x) \Big| & & \Big| \scriptstyle \textit{whisk-L}(x,q) \\
v \cdot x & & x \cdot v \\
\scriptstyle \textit{whisk-L}(v,p) \Big| & & \Big| \scriptstyle \textit{whisk-R}(p,v) \\
v \cdot y & \underset{EH(v,y)}{\rule[0.5ex]{4em}{0.4pt}} & y \cdot v
\end{array}
$$

Syllepsis: The Square

We use the naturality of Eckmann-Hilton:

$$
\begin{array}{ccc}
u \cdot x & \overset{EH(u,x)}{\rule{3cm}{0.4pt}} & x \cdot u \\
{\scriptstyle \textit{whisk-R}(q,x)} \Big| & & \Big| {\scriptstyle \textit{whisk-L}(x,q)} \\
v \cdot x & \overset{EH(v,x)}{\rule{3cm}{0.4pt}} & x \cdot v \\
{\scriptstyle \textit{whisk-L}(v,p)} \Big| & & \Big| {\scriptstyle \textit{whisk-R}(p,v)} \\
v \cdot y & \underset{EH(v,y)}{\rule{3cm}{0.4pt}} & y \cdot v
\end{array}
$$

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Future Directions

Where to go next:

- Use the syllepsis term to compute the Brunerie number, *i.e.*, prove that $\pi_4(S^3)$ is 2.
- Adapt the techniques from this proof to further open problems in synthetic homotopy type theory.
- Suggestions here: ...