Dependently typed algebraic theories and their homotopy algebras

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(Higher) Algebra in space theory

"Space theory" (HoTT) is dependently typed.

$$A{:}\mathsf{U}, x, y{:}A \vdash \mathrm{Id}_A(x, y) : \mathsf{U}$$

So algebra in space theory should use the full expressive power of dependent types.

Thesis

*The "natural" generalisations of multisorted algebraic theories from set theory to space theory are **dependently typed algebraic theories**.*

Question 1

What is a **dependently typed algebraic theory**?

Follow-up question

Question 2

What is a **space-valued model** of a dependently typed algebraic theory?

Question 2 (reframed for this talk)

For a dependently typed algebraic theory $\mathbf{T}$, is there a model category that presents the $(\infty,1)$-category of $\mathbf{T}$-models in spaces?

Many answers to Q1

- Cartmell's generalised algebraic theories
- Makkai's FOLDS vocabularies and theories
- Fiore's Σ_n-models with substitution
- Palmgren's DFOL signatures
- Others (Aczel, Belo, QIITs ...)

I'll use a strictly less general definition[1] than each of these, but one that:

- is Morita equivalent to GATs,
- that has a nice algebraic description,
- and a nice homotopy theory of models in spaces.

[1] These will be exactly Fiore's Σ_0-models with substitution.

Fewer answers to Q2

(Kapulkin–Szumiło[2]) & (Kapulkin–LeFanu Lumsdaine[3]):
The space-valued models of a dependently typed algebraic theory form a locally finitely presentable ∞-category.

This construction is very general but somewhat unwieldy: it results in a quasicategory, but type theory is usually interpreted in a model category.

Is there a direct way to get from a syntactic presentation of a dependently typed algebraic theory to a combinatorial model category of its "models in spaces"?

[2] [KS17]
[3] [KL16] ("Homotopy theory of type theories")

Caveat: ~~(higher)~~ algebra in space theory

The theories in this talk are discrete dependently typed algebraic theories.

Just as

- ordinary multisorted algebraic theories (1-categories with finite products) are discrete ∞-categories with finite products,
- and Set-operads are discrete ∞-operads.

Hope: Adding identity types gives all non-discrete dependently typed algebraic theories.

Introduction

Dependently typed algebraic theories

Models and homotopy models of $\mathbb{C}$-contextual categories

Rigidification of homotopy algebras (jwipw S. Henry)

Future work

Today's definition

A **dependently typed algebraic theory** is the data of:

- A type signature $\mathbb{C}$,
- and a $\mathbb{C}$-typed theory.

Type signatures

A **type signature** is a small category C that is

1. direct ($\exists$ an identity-reflecting functor $\mathrm{C} \to \lambda$ to some ordinal),
2. and "locally finite": its slice categories are finite (every pullback $B = 1 \times_{\mathrm{C}} \mathrm{C}^{\to}$ as below in the 1-category Cat is a finite category).

$$\begin{array}{ccc} B & \longrightarrow & \mathrm{C}^{\to} \\ \downarrow & \lrcorner & \downarrow \mathrm{t} \\ 1 & \longrightarrow & \mathrm{C} \end{array}$$

Type signatures = "locally finite", direct categories (lfd categories).

C-typed theories

Let C be a type signature. A C-**typed theory** is a finitary[4] monad on the presheaf category $\widehat{\mathrm{C}} = [\mathrm{C}^{op}, \mathrm{Set}]$.

[4] One whose endofunctor preserves filtered colimits.

Recall

A **multisorted algebraic theory** is the data of:

- A set S (of sorts),
- and a finitary monad T on $\widehat{S} = \mathrm{Set}_{/S}$ (the S-sorted theory).

Rmk: Any set is a discrete (and hence lfd) category.

Type dependence $\sim$ Cellularity

These definitions are based on a duality between cellular structures and type dependency.

$$\vdash V \text{ type} \qquad \mathrm{x}{:}V, \mathrm{y}{:}V \vdash E(\mathrm{x},\mathrm{y}) \text{ type}$$

…is a graph (0-cells = nodes, 1-cells = arrows).

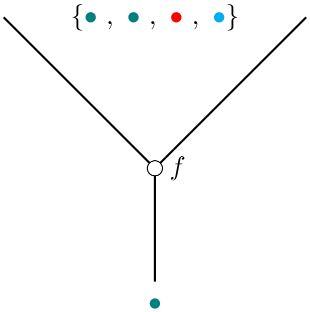
$$\vdash X_0 \qquad \mathrm{x},\mathrm{y}{:}X_0 \vdash X_1(\mathrm{x},\mathrm{y})$$

$$\mathrm{x},\mathrm{y},\mathrm{z}{:}X_0, \mathrm{f}{:}X_1(\mathrm{x},\mathrm{y}), \mathrm{g}{:}X_1(\mathrm{y},\mathrm{z}), \mathrm{h}{:}X_1(\mathrm{x},\mathrm{z}) \vdash X_2(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{f},\mathrm{g},\mathrm{h})$$

…is a $\boldsymbol{\Delta}'_{\leq 2}$-type (a 2-truncated semisimplicial type).

Type dependence ∼ Cellularity

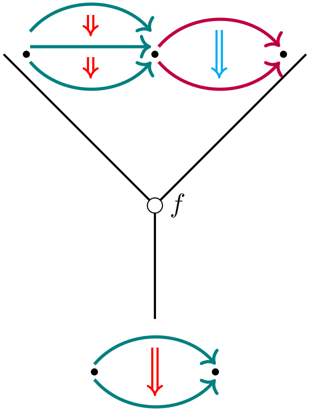
Multisorted algebraic theories are cartesian multicategories:



An operation/multimorphism takes a finite coproduct of points as input, and outputs a point.

Type dependence ∼ Cellularity

Dependently typed algebraic theories are **cellular** cartesian multicategories.



An operation/multimorphism takes a finite cell complex as input, and outputs a cell.

Intuition

C is an **inverse** category if C^{op} is direct.

- Objects of a direct category represent "cells" of some "shape" and morphisms are subcell inclusions.
- Objects of an inverse category are "dependent types" and morphisms are type dependencies.
- Local finiteness says that
  1. every type/cell of the signature is of finite dimension,
  2. and every type/cell of the signature depends on a finite context of variables (has finitely many subcells).

Examples of type signatures

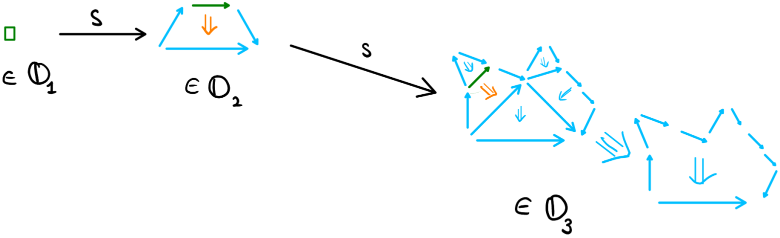
1. Any set S (seen as a discrete category).
2. The ordinal ω (seen as a totally ordered poset).
3. The category $\mathbb{G}$ of *globes* :

$$D^0 \underset{t}{\overset{s}{\rightrightarrows}} D^1 \underset{t}{\overset{s}{\rightrightarrows}} D^2 \underset{t}{\overset{s}{\rightrightarrows}} \dots \quad ; \quad ss = ts \ , \ st = tt$$

4. The category $\mathbb{G}_{\leq n} \subset \mathbb{G}$.

Examples of type signatures

5. The category $\mathbb{O}$ of *opetopes* :



Examples of type signatures

6. A Reedy category R has a wide direct subcategory R'. In many examples, R' is lfd:
   - $\mathbf{\Delta}'$ = the *semi-simplex* category,
   - $\mathbf{\Omega}'_p$ = category of *planar semi-dendrices*,
   - $R = \mathbf{\Theta}$, Joyal's cell category .

   (in each case R' is the wide subcategory of monos.)
7. If C is lfd, then for every $X\colon \mathrm{C}^{op} \to \mathrm{Set}$, the category of elements C/X is lfd.

The contextual category of cell complexes

$\mathrm{Cell}_{\mathrm{C}}$ has a graded set of objects $\mathrm{ob}(\mathrm{Cell}_{\mathrm{C}}) \stackrel{\text{def}}{=} \coprod_{n\in\mathbb{N}} (\mathrm{Cell}_{\mathrm{C}})_n$

- $(\mathrm{Cell}_{\mathrm{C}})_0$ consists of the empty presheaf $\emptyset \in \widehat{\mathrm{C}}$,
- for $\emptyset \to \ldots X$ in $(\mathrm{Cell}_{\mathrm{C}})_n$, c in C and $c \hookleftarrow \partial c \to X$ in $\widehat{\mathrm{C}}$, we make a *choice* of pushout square

$$\begin{array}{ccc} \partial c & \longrightarrow & X \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & Y \end{array}$$

giving $\emptyset \to \ldots \to X \to Y$ in $(\mathrm{Cell}_{\mathrm{C}})_{n+1}$.

We define $\mathrm{Cell}_{\mathrm{C}}(\emptyset \to \ldots X, \emptyset \to \ldots Y) \stackrel{\text{def}}{=} \widehat{\mathrm{C}}(X, Y)$.

The free C-typed algebraic theory

Fact

$\mathrm{Cx(C)} \stackrel{\text{def}}{=} \mathrm{Cell}_{\mathrm{C}}^{op}$ is the free contextual category on C (its *syntactic category*).

Precisely, for any contextual category D, morphisms $\mathrm{Cx(C)} \to \mathsf{D}$ correspond to contextual functors $\mathrm{C}^{op} \to \mathsf{D}$.

A functor $F\colon \mathrm{C}^{op} \to \mathsf{D}$ is **contextual** if

- for c in C, Fc is in D_k where $k = |\mathrm{ob}(\mathrm{C}_{/c})|$,
- and the "parent" projection $Fc \to \mathrm{ft}(Fc)$ in D is a morphism of limits corresponding to $\mathrm{C}/\partial c \hookrightarrow \mathrm{C}_{/c}$.

Example

A semisimplicial type in D is a contextual functor $\mathbf{\Delta}'^{op} \to \mathsf{D}$.

Finitary monads on $\widehat{\mathrm{C}}$

Since C is lfd, $\mathrm{Cell}_{\mathrm{C}}$ is a completion of C under finite colimits.

A C-**theory** is an identity-on-objects, finitely cocontinuous functor $\mathrm{Cell}_{\mathrm{C}} \to \Theta$. A morphism of C-theories is a triangle $\mathrm{Cell}_{\mathrm{C}} \to \Theta \to \Theta'$.

Fact

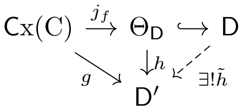
The category of C-theories is equivalent to the category of finitary monads on $\widehat{\mathrm{C}}$ (and monad morphisms).

Dependently typed algebraic theories

A C-**contextual category** is a morphism $f\colon \mathsf{Cx}(\mathrm{C}) \to \mathsf{D}$ in CxlCat whose (id.-on-objects, f.f.) factorisation

$$\mathsf{Cx}(\mathrm{C}) \xrightarrow{j_f} \Theta_{\mathsf{D}} \hookrightarrow \mathsf{D}$$

is such that for every diagram



where g is in CxlCat and h is any functor, $\exists! \tilde{h}$ in CxlCat making the diagram commute.

A morphism of C-contextual categories is a triangle $\mathsf{Cx}(\mathrm{C}) \to \mathsf{D} \to \mathsf{D}'$ in CxlCat.

Classification of dependently sorted algebraic theories

Theorem (LS–LeFanu Lumsdaine)

Given a type signature C*, the categories*

1. $\mathrm{FinMnd}(\widehat{\mathrm{C}})$ *of finitary monads on* $\widehat{\mathrm{C}}$*,*
2. $\mathrm{Law}_{\mathrm{C}}$ *of* C*-theories,*
3. *and* $\mathrm{CxlCat}_{\mathrm{C}}$ *of* C*-contextual categories,*

are equivalent.

Examples of dependently typed algebraic theories

Many well-known finitary monads are dependently typed algebraic theories.

1. For $S \in \mathrm{Set}$, every S-sorted algebraic theory.
2. The identity monads on $\widehat{\mathbb{G}_1}$ (graphs), $\widehat{\mathbb{G}}$ (globular sets), $\widehat{\mathbb{O}}$ (opetopic sets), $\widehat{\boldsymbol{\Delta}'}$ (semi-simplicial sets).
3. The free-category monad on $\widehat{\mathbb{G}_1}$.
4. The free-strict-ω-category monad on $\widehat{\mathbb{G}}$.
5. For $T \colon \widehat{\mathrm{C}} \to \widehat{\mathrm{C}}$ a finitary cartesian monad, every Burroni–Leinster T-operad $T' \to T$ (e.g. globular operads).
6. Every free-*weak*-ω-category monad on $\widehat{\mathbb{G}}$ (for a Gr-coherator).

and many more...

Introduction

Dependently typed algebraic theories

Models and homotopy models of $\mathbb{C}$-contextual categories

Rigidification of homotopy algebras (jwipw S. Henry)

Future work

Discrete models of C-contextual categories

Definition

A **(Set-)model** of a C-contextual category $\mathsf{Cx}(\mathrm{C}) \to \mathsf{D}$ is a presheaf $X \colon \mathsf{D} \to \mathrm{Set}$ such that the composite $\mathsf{Cx}(\mathrm{C}) \to \mathsf{D} \xrightarrow{X} \mathrm{Set}$

1. takes $\emptyset \in \mathrm{Cell}_{\mathrm{C}}$ to $1 \in \mathrm{Set}$,
2. and takes every chosen pushout

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & \ulcorner & \big\downarrow \\ c & \longrightarrow & X_{n+1} \end{array}$$

   in $\mathrm{Cell}_{\mathrm{C}}$ to a pullback square in Set.

A morphism of models is just a natural transformation.

Discrete models and algebras

Models of a C-contextual category $\mathrm{Cx}(\mathrm{C}) \to \mathsf{D}$ are equivalently:

1. algebras of the associated finitary monad on $\widehat{\mathrm{C}}$,
2. Set-models of the underlying contextual category D.

Morita equivalence with EATs

Theorem

$\mathcal{C}$ is locally finitely presentable iff it is the category of models of a C-contextual category (for some type signature C).

Proof

One direction is obvious.

1. Every category of models of a C-contextual category is a category of models of a finite-limit sketch.

Conversely,

2. Consider the non-full inclusion $i_{\Delta'} \colon \mathbf{\Delta}' \to \mathrm{Cat}$. It has an associated *semisimplicial nerve functor* $N_{\mathbf{\Delta}'} \colon \mathrm{Cat} \to \widehat{\mathbf{\Delta}'}$

   For $A \in \mathrm{Cat}$, let $\mathbf{\Delta}'{\downarrow}A$ be the comma-category. Then $\mathbf{\Delta}'{\downarrow}A$ is the category of elements $\mathbf{\Delta}'/(N_{\mathbf{\Delta}'}A)$. Thus $\mathbf{\Delta}'{\downarrow}A$ is a type signature.

   There is an obvious functor $\tau_A \colon \mathbf{\Delta}'{\downarrow}A \to A$ taking $\{0 < \dots n\} \xrightarrow{f} A$ to $f(n)$.

   (Cisinski[5]) The pullback functor $\tau_A^* \colon \widehat{A} \hookrightarrow \widehat{\mathbf{\Delta}'{\downarrow}A}$ is fully faithful.

[5] [Cis03, Prop. 6.9]

3. Every locally finitely presentable category $\mathcal{C}$ has an ω-accessible, fully faithful right adjoint $\mathcal{C} \hookrightarrow \widehat{A}$ to a presheaf category. Then the composite

$$\mathcal{C} \hookrightarrow \widehat{A} \xhookrightarrow{\tau_A^*} \widehat{\mathbf{\Delta}'{\downarrow}A}$$

is fully faithful, monadic and ω-accessible. So $\mathcal{C}$ is the category of algebras of a finitary (idempotent) monad on $\widehat{\mathbf{\Delta}'{\downarrow}A}$. $\square$

Models in spaces of multisorted algebraic theories

Let S be a set and $\mathbf{T}$ be an S-sorted algebraic theory.

A simplicial $\mathbf{T}$-algebra is a finite-product-preserving functor $F\colon \mathbf{T} \to \mathrm{sSet}$ (equivalently, a simplicial diagram $F\colon \Delta^{op} \to \mathbf{T}\text{-}\mathrm{Mod}$).

A homotopy model of $\mathbf{T}$ is a functor $F\colon \mathbf{T} \to \mathrm{sSet}$ taking finite products to homotopy limits.

Remark

All products in sSet are homotopy limits.

So F is a homotopy $\mathbf{T}$-algebra if every $F(s_1 \times \ldots \times s_k) \to Fs_1 \times \ldots \times Fs_k$ is a weak equivalence in sSet.

Models in spaces of C-contextual categories

Let C be a type signature.

Definition

A **homotopical** C-**space** is a simplicial presheaf $F\colon \mathrm{Cell}_{\mathrm{C}}^{op} \to \mathrm{sSet}$

1. such that $F\emptyset$ is contractible,
2. and F takes every chosen pushout

$$\begin{array}{ccc} \partial c & \longrightarrow & X_n \\ \big\downarrow & & \big\downarrow \\ c & \longrightarrow & X_{n+1} \end{array}$$

to a homotopy pullback square, i.e. $FX_{n+1} \simeq FX_n \times^h_{F\partial c} Fc$.

Models in spaces of $\mathbb{C}$-contextual categories

Definition

A **homotopical model** of a $\mathbb{C}$-contextual category $\mathsf{Cx}(\mathbb{C}) \to \mathsf{D}$ is a simplicial presheaf $\mathsf{D} \to \mathrm{sSet}$ such that $\mathsf{Cx}(\mathbb{C}) \to \mathsf{D} \to \mathrm{sSet}$ is a homotopical $\mathbb{C}$-space.

Remark

Pullbacks in sSet are not homotopy limits, so we cannot reformulate this condition by requiring that the canonical map $FX_{n+1} \to FX_n \times_{F\partial c} Fc$ to the strict pullback be a weak equivalence.

Flasque model structure

Due to this subtlety, we introduce an intermediate global model structure on the simplicial presheaf category $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}}) \stackrel{\text{def}}{=} [\mathrm{Cell}_{\mathrm{C}}^{op}, \mathrm{sSet}]$.

Flasque boundaries

For $c \in \mathrm{C}$, let "∂c" be the colimit of the composite

$$\mathrm{C}_{/c}^{-} \to \mathrm{C} \hookrightarrow \mathrm{Cell}_{\mathrm{C}} \hookrightarrow \widehat{\mathrm{Cell}_{\mathrm{C}}}.$$

We have a composite inclusion in $\widehat{\mathrm{Cell}_{\mathrm{C}}}$

$$\text{"}\delta_c\text{"} \colon \text{"}\partial c\text{"} \longhookrightarrow \partial c \xhookrightarrow{\delta_c} c$$

where $\partial c \hookrightarrow c$ is representable in $\mathrm{Cell}_{\mathrm{C}}$.

Definition

A map $p\colon X \to Y$ in $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})$ is a ∂-**flasque fibration** if the "pullback-hom" map

$$\langle\text{``}\delta_c\text{''}, p\rangle : X_c \longrightarrow \mathrm{Map}(\text{``}\partial c\text{''}, X) \times_{\mathrm{Map}(\text{``}\partial c\text{''}, Y)} Y_c$$

in sSet is a Kan fibration.

Fact

The **flasque** model structure on $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})$ whose weak equivalences are the global (objectwise) weak equivalences, and whose fibrations are the ∂-flasque fibrations, exists. We write it $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_\partial$.

Remarks

1. $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\partial}$ is *intermediate*: the identity functor gives Quillen equivalences

$$\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\mathit{proj}} \rightleftarrows \mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\partial} \rightleftarrows \mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\mathit{inj}}$$

   where (*proj* = projective) and (*inj* = injective) model structures.

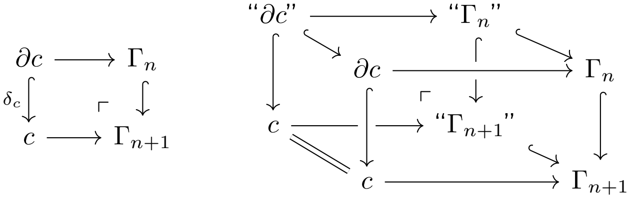
2. For the inclusion $i\colon \mathrm{C} \hookrightarrow \mathrm{Cell}_{\mathrm{C}}$, both adjunctions

$$\mathrm{Sp}(\mathrm{C})_{\mathit{inj}} \underset{i_*}{\overset{i_!}{\underset{\longleftarrow i^* \longrightarrow}{\hookrightarrow}}} \mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\partial}$$

   are Quillen for the injective Reedy model structure $\mathrm{Sp}(\mathrm{C})_{\mathit{inj}}$.

Model structure for homotopy C-spaces

For every object of $\mathrm{Cell}_{\mathrm{C}}$ (a finite cell complex $\emptyset \to \Gamma_1 \to \ldots \to \Gamma$) we inductively define the subrepresentable "Γ" $\hookrightarrow \Gamma$ in $\widehat{\mathrm{Cell}_{\mathrm{C}}}$, by defining "$\emptyset$" to be the empty presheaf and by:



Definition

The model structure for homotopy C-spaces is the left Bousfield localisation of $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\partial}$ at the set of maps (between cofibrant objects)

$$S_{\partial} \overset{\text{def}}{=} \{s_{\Gamma} \colon \text{``}\Gamma\text{''} \hookrightarrow \Gamma \mid \Gamma \in \mathrm{Cell}_{\mathrm{C}}\}.$$

We write it as $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^{l}_{\partial}$.

Fibrant objects of $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^{l}_{\partial}$ are called **homotopy** C-**spaces**.

Recall

X is a fibrant object of $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^{l}_{\partial}$ iff it is S_{∂}-**local** : i.e. it is fibrant in $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})_{\partial}$ and every $\langle s_{\Gamma}, X\rangle : X_{\Gamma} \to \mathrm{Map}(\text{``}\Gamma\text{''}, X)$ is a weak equivalence in sSet.

Fact

The adjunction $i^* : \mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^l_\partial \rightleftarrows \mathrm{Sp}(\mathrm{C})_{inj} : i_*$ is a Quillen equivalence.

Thus $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^l_\partial$ presents the presheaf ∞-category $\mathcal{P}(\mathrm{C})$.

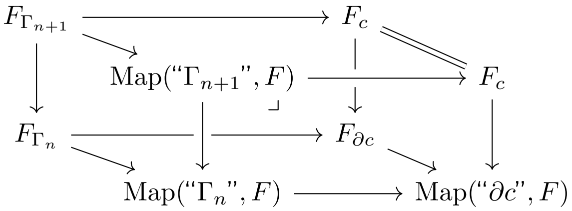
Theorem

If F is fibrant in $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^l_\partial$ (a homotopy C-space), then it is a homotopical C-space.

Proof

F is S_∂-local, so $F_\emptyset \to \mathrm{Map}(\text{“}\emptyset\text{”}, F) = 1$ is a weak equivalence.

We have the cube in sSet whose front face is cartesian.



- All corners of the cube are fibrant objects,
- $F_c \to \mathrm{Map}(\text{“}\partial c\text{”}, F)$ is a Kan fibration,
- so the front face is a homotopy pullback.

The intervening arrows are weak equivalences, so the back face is a homotopy pullback. □

Homotopy models of any C-contextual category

$\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^{l}_{\partial}$ is the model structure for homotopy models of the **initial** C-contextual category.

For an arbitrary C-contextual category $\mathrm{Cx}(\mathrm{C}) \to \mathsf{D}$, we consider the (id-on-objects, f.f.) factorisation $\mathrm{Cell}_{\mathrm{C}} \xrightarrow{j} \Theta_{\mathsf{D}} \hookrightarrow \mathsf{D}^{op}$.

There is a model structure for homotopy D-algebras on $\mathrm{Sp}(\mathsf{D}^{op})$ whose fibrant objects are homotopical models of D.

Rigidification

Is every homotopy D-algebra equivalent to a simplicial D-algebra?

Rigidification for multisorted algebraic theories

Let S be a set and $\mathbf{T}$ be an S-sorted algebraic theory.

The projective, Reedy and injective model structures on $\mathrm{Sp}S = \mathrm{sSet}^S$ coincide.

(Quillen[6]) There is a "transferred" model structure on the category $\mathrm{s}\mathbf{T}\text{-}\mathrm{Alg}$ of simplicial $\mathbf{T}$-algebras. Its fibrations and weak equivalences are created by the monadic functor $\mathrm{s}\mathbf{T}\text{-}\mathrm{Alg} \to \mathrm{sSet}^S$.

The reflective adjunction $\mathrm{Sp}(\mathbf{T}^{op})_{proj} \rightleftarrows \mathrm{s}\mathbf{T}\text{-}\mathrm{Alg}$ is a Quillen adjunction.

[6] [Qui67, II.4], [Ber06, Th. 4.7]

The model structure $\mathrm{Sp}(\mathrm{Cell}_S)_\partial$ is just the projective model structure.

Write the free functor as $j\colon \mathrm{Cell}_S \to \mathbf{T}^{op}$.

We can left Bousfield localise $\mathrm{Sp}(\mathrm{Cell}_S)_{proj}$ and $\mathrm{Sp}(\mathbf{T}^{op})_{proj}$ at the sets of maps S_∂ and $j_! S_\partial$ respectively.

The Bousfield localisation $\mathrm{Sp}(\mathbf{T}^{op})^l$ is the model structure for homotopy $\mathbf{T}$-algebras.

We have an exact adjoint square

$$\begin{array}{ccc} \mathrm{sSet}^S & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathrm{s}\mathbf{T}\text{-Alg} \\ {\scriptstyle i^*}\uparrow\downarrow{\scriptstyle i_*} & & {\scriptstyle h}\uparrow\downarrow{\scriptstyle N} \\ \mathrm{Sp}(\mathrm{Cell}_S)^l_{\mathit{proj}} & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathrm{Sp}(\mathbf{T}^{op})^l_{\mathit{proj}} \end{array}$$

in which the left vertical adjunction is a Quillen equivalence, and the horizontal adjunctions are Quillen.

Theorem (Badzioch, Bergner[7])

The right vertical adjunction is a Quillen equivalence.

[7] [Bad02, Ber06]

Rigidification of homotopy $\mathrm{Cx}(\mathrm{C})$-algebras

Let C be a type signature, so $\widehat{\mathrm{C}}$ is the category of Set-models of $\mathrm{Cx}(\mathrm{C})$.

Consider the weak factorisation system on $\widehat{\mathrm{C}}$ generated by the set $I = \{\partial c \hookrightarrow c \mid c \in \mathrm{C}\}$ of boundary inclusions. This is the WFS $(\mathrm{mono}, (\mathrm{mono})^{\pitchfork})$.[8]

Along with the FS $(\mathrm{iso}, \mathrm{all})$, this defines a **combinatorial premodel structure** [Bar19] on $\widehat{\mathrm{C}}$ whose cofibrations are the monomorphisms.

The algebra of CPM categories ensures that the tensor product of locally presentable categories $\widehat{C} \otimes \widehat{\mathbf{\Delta}} = \mathrm{SpC}$ is a CPM category. This premodel structure on SpC is exactly the Reedy (=injective) model structure.

We can see the Quillen equivalence $\mathrm{Sp}(\mathrm{Cell}_{\mathrm{C}})^{l} \rightleftarrows \mathrm{SpC}_{inj}$ as a rigidification theorem for homotopy $\mathrm{Cx}(\mathrm{C})$-algebras.

[8] [Mak95] calls maps in $(\mathrm{mono})^{\pitchfork}$ *fiberwise surjective*.

Rigidification of homotopy D-algebras

Let $\mathrm{Cx}(\mathrm{C}) \to \mathsf{D}$ be a C-contextual category, and let I_{D} be the image of I in $\mathsf{D}^{op} \subset \mathsf{D}\text{-}\mathrm{Mod}$.

Mutatis mutandis, there is a CPM structure on the category $\mathrm{s}\mathsf{D}\text{-}\mathrm{Mod}$ of simplicial D-algebras.

It is moreover a **weak model structure** in the sense of [Hen20], and is the weak model structure transferred along the monadic functor $\mathrm{s}\mathsf{D}\text{-}\mathrm{Mod} \to \mathrm{SpC}_{inj}$.

Rigidification of homotopy D-algebras

We can left Bousfield localise the projective model structure on $\mathrm{Sp}(\mathsf{D}^{op})$ at the set of maps $S_{\mathsf{D}} = \{\text{``}\Gamma\text{''} \to \Gamma \mid \Gamma \in \mathsf{D}\}$.

Theorem (Rigidification for homotopy D-algebras)

The adjunction $\mathrm{Sp}(\mathsf{D}^{op})^l \rightleftarrows \mathrm{sD\text{-}Mod}$ *is a weak Quillen equivalence.*

What I'm thinking about

1. Dependently coloured operads :
   Coloured operads $\rightsquigarrow$ algebraic theories
   vs. ??? $\rightsquigarrow$ dependently typed algebraic theories.
2. *Polygraphs* (contexts) of a $\mathbb{C}$-contextual category D, and their relation to generic-free factorisations in D.
3. Rezk completion and univalent algebras: Homotopy D-algebras are *à la* Segal spaces, so what about *complete* Segal spaces? (jwipw M. Shulman)

Thank you!

Bernard Badzioch.
Algebraic theories in homotopy theory.
Annals of Mathematics, 155(3):895–913, 2002.

Reid William Barton.
A model 2-category of enriched combinatorial premodel categories.
PhD thesis, Harvard University, 2019.

Julia E Bergner.
Rigidification of algebras over multi-sorted theories.
Algebraic & Geometric Topology, 6(4):1925–1955, 2006.

J. W. Cartmell.
Generalised algebraic theories and contextual categories.
PhD thesis, University of Oxford, 1978.

Denis-Charles Cisinski.
Images directes cohomologiques dans les catégories de modèles.
Annales Mathématiques Blaise Pascal, 10(2):195–244, 2003.

Marcelo Fiore.
Second-order and dependently-sorted abstract syntax.
In *2008 23rd Annual IEEE Symposium on Logic in Computer Science*, pages 57–68. IEEE, 2008.

Simon Henry.
Combinatorial and accessible weak model categories.
arXiv preprint arXiv:2005.02360, 2020.

Chris Kapulkin and Peter LeFanu Lumsdaine.
The homotopy theory of type theories.
arXiv preprint arXiv:1610.00037, 2016.

Chris Kapulkin and Karol Szumiło.
Internal languages of finitely complete $(\infty, 1)$-categories, 2017.

Tom Leinster.
Higher Operads, Higher Categories.
Cambridge University Press, 2004.

Michael Makkai.

First Order Logic with Dependent Sorts, with Applications to Category Theory.
Preprint, 1995.
math.mcgill.ca/makkai/folds/foldsinpdf/FOLDS.pdf.

Daniel G. Quillen.
Homotopical Algebra.
Springer-Verlag, 1967.