Choice, Collection and Covering in Cubical Sets

Andrew W Swan

CMU

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Definition

Brouwer's principle states that all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

Explicitly, for all $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and all $\alpha : \mathbb{N} \to \mathbb{N}$, there merely exists N such that for all $\beta : \mathbb{N} \to \mathbb{N}$, if $\beta(n) = \alpha(n)$ for $n < N$, then $F(\alpha) = F(\beta)$.

Theorem (S.)

Working in a metatheory where Brouwer's principle holds, all of the following are false in cubical sets.
Choice principles:

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2. *A weak form of countable choice due to Bridges, Richman and Schuster.*
3. $\mathbf{AC}(\mathbb{N}, 2)$

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Collection principles:

1. *Set theoretic and type theoretic versions of collection*
2. *Set theoretic and type theoretic versions of fullness*
3. *Weakly initial set of covers*

Definition

We say a map $f : B \to A$ is a *surjection* if every fibre is merely inhabited: $\prod_{a:A} \|\mathbf{hFibre}_f(a)\|$. We will also say that the pair (B, f) is a *cover* of A.

We say A is *projective* if every cover of A merely has a section.

The axiom of *countable choice* states that $\mathbb{N}$ is projective.

Definition (Escardó)

Suppose we are given a binary sequence $\alpha\colon \mathbb{N} \to 2$. We write $\langle\alpha\rangle$ for the type $\sum_{n:\mathbb{N}} \alpha(n) = 1$.
We define $\mathbb{N}_\infty$ to be the collection of binary sequences $\alpha\colon \mathbb{N} \to 2$ such that $\langle\alpha\rangle$ is an hProposition.

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Proposition

Suppose that $F \colon \mathbb{N}_\infty \to 2$ is continuous. Then there is some N such that that $F(\underline{n}) = F(\infty)$ for $n > N$.

Definition (Escardó-Knapp)

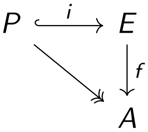
The *Escardó-Knapp axiom of choice* **EKC** states that for all $\alpha : \mathbb{N}_\infty$, $\langle \alpha \rangle$ is projective.

1. Since $\langle \alpha \rangle$ is an hProposition and 0 and 1 are both projective, **EKC** follows from the law of excluded middle. In fact **LPO** is sufficient.
2. **EKC** follows from countable choice. In fact a weak form of countable choice due to Bridges, Richman and Schuster is sufficient.

Suppose we are given a family of types $B \colon A \to \mathcal{U}$. A *multi valued section* of B consists of a family of hPropositions $P \colon (a : A) \to Ba \to \textbf{hProp}$ together with a proof of $\prod_{a:A} \exists_{b:B} Pab$. One can also use the more diagramatic version of the definition from predicative algebraic set theory:

Definition (Van den Berg, Moerdijk)

A multi valued section of a map $f \colon E \to A$ consists of an embedding $i \colon P \hookrightarrow E$ such that the composition $f \circ i$ is surjective:

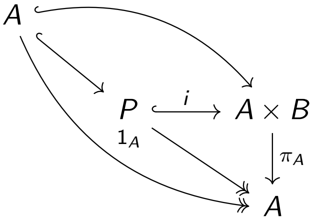


Definition

A *multivalued function* from A to B is a multivalued section of the constant family of types $\lambda a.B$.
Diagramatically, it is a multivalued section of the projection map $\pi_A \colon A \times B \to A$:

$$\begin{array}{ccc} P & \overset{i}{\hookrightarrow} & A \times B \\ & \searrow\!\!\!\!\searrow & \downarrow{\scriptstyle \pi_A} \\ & & A \end{array}$$

The axiom **AC**(A, B) states that every multivalued section can be refined by a single valued section:



AC$(\mathbb{N}, 2)$ holds in most realizability models, *even if it does not hold in the metatheory*. It also follows from the law of excluded middle.

Definition

We say $X : \mathcal{U}_0$ is *covered by a set* if there exists a cover $Y \twoheadrightarrow X$ where Y is an hSet. We say *sets cover* if every type $X : \mathcal{U}_0$ is covered by an hSet.

Theorem

Suppose that countable choice holds. Then $\prod_{\mathbb{N}} \mathbb{S}^1$ *is covered by an hSet.*

Proof.

We have a constant map $1 \to \prod_{\mathbb{N}} \mathbb{S}^1$ given by $\lambda x, n.\mathtt{base}$. It suffices to show this map is surjective. Let $f : \mathbb{N} \to \mathbb{S}^1$. We need to find an element of $\| \prod_{n:\mathbb{N}} fn = \mathtt{base} \|$. Since $\mathbb{S}^1$ is 1-truncated, $fn = \mathtt{base}$ is an hSet for all n, and since $\mathbb{S}^1$ is connected, we have an element of $\prod_{n:\mathbb{N}} \| fn = \mathtt{base} \|$, and so we can apply countable choice. $\square$

Collection principles are used to show that the failure of the axiom of choice does not cause "size issues."
We will consider the following collection principles:

1. Collection
2. Fullness
3. Weakly initial set of covers (**WISC**)[1]

We will state them using the notion of weak initiality:

Definition

Let $\mathbb{C}$ be a category and let $X\colon I \to \mathrm{Ob}(\mathbb{C})$. We say X is *weakly initial* if for every object Y of $\mathbb{C}$ there merely exists $i : I$ and a morphism $X_i \to Y$.

[1] **WISC** is usually considered a choice axiom.

Definition

The axiom of *collection* states that for every X the inclusion map $\mathrm{Cov}_0(X) \hookrightarrow \mathrm{Cov}_1(X)$ is weakly initial.

Definition

The axiom *weakly initial set of covers*, **WISC**, states that there merely exists $I : \mathcal{U}_0$ together with a weakly initial map $I \to \mathrm{Cov}_0(X)$.

Proposition

$\mathbf{AC}(B)$ *is true if and only if the canonical map* $\prod_{a:A} B(a) \to \mathrm{mvs}(B)$ *is weakly initial.*

Definition

The axiom of *fullness* for maps $E \to A$ states that there merely a type I (WLOG an hSet) and a weakly initial map $I \to \mathrm{mvs}(B)$.

1. Fullness follows from $\mathbf{AC}(E)$.
2. Fullness follows from propositional resizing.
3. It allows us to deal with "size issues" caused by working in a setting where both choice and propositional resizing fail. E.g. In constructive set theory it is used to show the class of Dedekind reals is a set.

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Van den Berg, Moerdijk: (Algebraic set theory formulations of) WISC, fullness and collection are preserved by “typical” topos theoretic constructions: sheaf toposes, realizability toposes, slice toposes.

Independence results for WISC, fullness and collection in set theory are possible but require sophisticated techniques.

Theorem (Van den Berg 2012, Karagila)

WISC *is independent of* **ZF**.

Van den Berg derived this as a corollary of a sophisticated result due to Gitik using forcing and large cardinals. Karagila showed the large cardinal assumption can be removed using class forcing.

Theorem (Friedman-Ščedrov 1985)

Collection is independent of $\mathbf{IZF_{Rep}}$.

Friedman and Ščedrov's proof uses forcing and a clever Kripke model.

Theorem (Lubarsky 2006)

Fullness is independent of $\mathbf{CZF_{Exp}}$.

Lubarksy developed a new kind of forcing for this result called *forcing with settling*. It can also be proved using realizability (S).

Definition (Cohen, Coquand, Huber, Mörtberg)

The *cube category* is the category where $\mathbb{N}$ is the set of objects and a morphism from m to n is a homomorphism from the free De Morgan algebra on m elements to the free De Morgan algebra on n elements. A *cubical set* is a functor from the cube category to sets.

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Cubical sets form a constructive model of homotopy type theory.

In the interpretation of extensional type theory in a locally cartesian closed category:

- Types in context Γ are interpreted as maps $A \to \Gamma$.
- Terms are interpreted as sections $\Gamma \to A$ (we will also refer to sections as *points*).
- Two terms are propositionally equal only if they are equal.
- Hence if a type is an hproposition it has at most one section.
- Propositional truncation "strictly identifies points."

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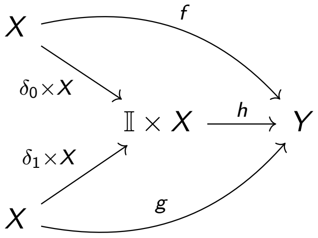
- Hence if a type is an hproposition it has at most one section.
- Propositional truncation "strictly identifies points."

In CCHM cubical sets (like with many other homotopical models) we make use of an interval object $\delta_0, \delta_1 \colon 1 \rightrightarrows \mathbb{I}$.

Definition

A *point* of a cubical set X, is a map $x \colon 1 \to X$.
A *path* in a cubical set X is a map $p \colon \mathbb{I} \to X$.
A *homotopy* between two maps $f, g \colon X \to Y$ is a map h such that the diagram below commutes:



In cubical sets:

- Types in context Γ are interpreted as maps $A \to \Gamma$ together with "fibration structure".
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- An hProposition can have many different sections as long as any two are homotopic.
- Proposition truncation keeps points separate, but adds new paths between them.

This observation was also used in a previous result due to S. and Uemura: Church's thesis does not hold in cubical assemblies.

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Coquand, Huber and Mörtberg construct propositional truncation for a family of types X in context Γ by an inductive definition:

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3. Whenever $\phi \in \mathbb{F}(A) \setminus \{\top\}$, and u is a ϕ-open box over γ in $\|X\|$, $\|X\|(A, \gamma)$ contains an element $\mathtt{hcomp}(\phi, u)$.

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We can also consider the inductive definition obtained by removing the $\mathtt{sq}$ elements, which we refer to as *local fibrant replacement*:

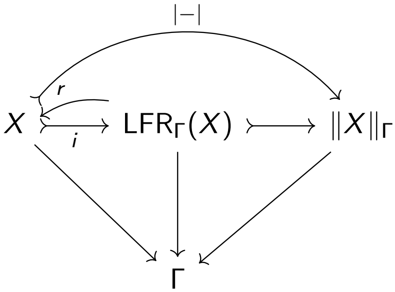
1. For each $x \in X(A, \gamma)$, $\mathsf{LFR}(X)(A, \gamma)$ contains an element $|x|$.
2. Whenever $\phi \in \mathbb{F}(A) \setminus \{\top\}$, and u is a ϕ-open box over γ in $\mathsf{LFR}(X)$, $\mathsf{LFR}(X)(A, \gamma)$ contains an element $\mathtt{hcomp}(\phi, u)$.

We can clearly factor the map $| - | \colon X \to \|X\|$ as two monomorphisms $X \rightarrowtail \mathsf{LFR}(X) \rightarrowtail \|X\|$ over Γ.

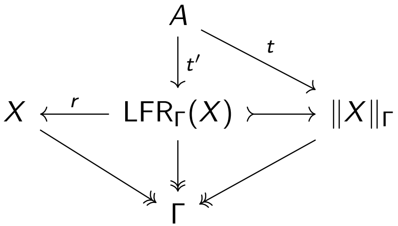
$\|X\|$ and $\mathsf{LFR}(X)$ have the following key properties:

1. $\mathsf{LFR}(X)$ is equivalent to X.
2. $\mathsf{LFR}(X)$ is a *locally decidable* subobject of $\|X\|$ i.e. for every A in the cube category and $\gamma \in \Gamma(A)$, every element of $\|X\|(A, \gamma)$ either belongs to $\mathsf{LFR}(X)(A, \gamma)$ or does not.
3. Every point of $\|X\|$ belongs to $\mathsf{LFR}(X)$.

We can illustrate the key lemmas categorically as follows. Suppose we are given a fibration $f\colon X \to \Gamma$. Then we can define propositional truncation and local fibrant replacement in the slice category over Γ to get the diagram below:



We say a map $t\colon A \to \|X\|_\Gamma$ is *squash free* if it factors (necessarily uniquely) through the monomorphism $\mathrm{LFR}_\Gamma(X) \rightarrowtail \|X\|_\Gamma$.



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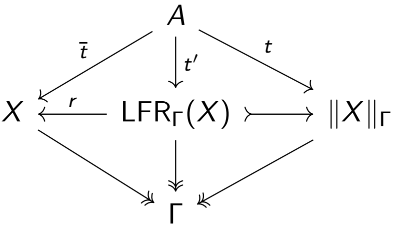

If t is squash free, we write the composition $r \circ t'$ as $\bar{t}$ and call this the *detruncation* of t.

Rephrasing two of the key lemmas, we get the following categorical versions:

1. Any map $1 \to \|X\|_{\Gamma}$ is squash free.
2. For any representable $\mathbf{y}A$, any map $\mathbf{y}A \to \|X\|_{\Gamma}$ is either sqaush free, or not.

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Also, by diagram chasing we get the following lemma.

Lemma

Suppose we are given maps $A \xrightarrow{h} A' \xrightarrow{t} \|X\|_\Gamma$. If t is squash free, then so is $t \circ h$, and we have $\overline{t \circ h} = \overline{t} \circ h$.

We apply this to paths $p\colon \mathbb{I} \to \|X\|_\Gamma$, noting that $\mathbb{I}$ is representable. There are many examples of such paths that are *not* squash free e.g. for any two points $x, y\colon 1 \to \|X\|_\Gamma$ we can use squash to define a path making a homotopy from x to y.
However, we have

1. Any path $p\colon \mathbb{I} \to \|X\|_\Gamma$ is either squash free, or not (even working constructively).
2. If p is degenerate, then it is squash free, and so $\bar{p}\colon \mathbb{I} \to X$ exists.
3. The endpoints $\delta_i \circ p$ are always squash free. When p is squash free we have $\delta_i \circ \bar{p} = \overline{\delta_i \circ p}$ for $i = 0, 1$.

Theorem

Assume Brouwer's principle. Then in the cubical set model of HoTT there is no surjection $f : B \to \prod_{\mathbb{N}} \mathbb{S}^1$ *where* B *is an hSet.*

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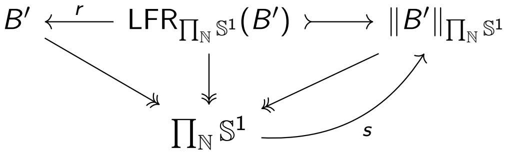
Assume Brouwer's principle. Then in the cubical set model of HoTT there is no surjection $f : B \to \prod_{\mathbb{N}} \mathbb{S}^1$ where B is an hSet.

First construct $B' := \sum_{z:\prod_{\mathbb{N}} \mathbb{S}^1} \mathbf{hFibre}_f(z)$, with $f' : B' \to \prod_{\mathbb{N}} \mathbb{S}^1$ the first projection. Then by the definition of surjection, the map $\|B'\|_{\prod_{\mathbb{N}} \mathbb{S}^1} \to \prod_{\mathbb{N}} \mathbb{S}^1$ has a section, giving us the map s in the diagram:

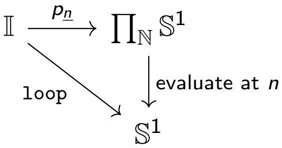


For each $\alpha : \mathbb{N}_\infty$, we define a map $p_\alpha : \mathbb{I} \to \prod_{\mathbb{N}} \mathbb{S}^1$ by

$$p_\alpha(i)(n) := \begin{cases} \mathtt{loop}(i) & \alpha(n) = 1 \\ \mathtt{base} & \text{otherwise} \end{cases}$$

Note that we constructed p_α to have the following properties:

- Setting $\alpha = \infty$, the path p_∞ is degenerate.
- Setting $\alpha = \underline{n}$ for $n : \mathbb{N}$ we have a commutative triangle below, where the right map is given by evaluation at n.



We consider the path $s \circ p_\alpha \colon \mathbb{I} \to \|B'\|$. It is either squash free, or not squash free, so we have a well defined function $F \colon \mathbb{N}_\infty \to 2$ defined by:

$$F(\alpha) := \begin{cases} 0 & s \circ p_\alpha \text{ is not squash free} \\ 1 & s \circ p_\alpha \text{ is squash free} \end{cases}$$

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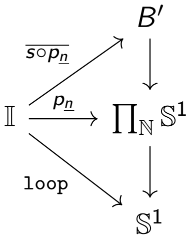
Since p_∞ is degenerate, so is $s \circ p_\infty$.

Degenerate paths are squash free, so we have $F(\infty) = 1$. By continuity, there is some $n : \mathbb{N}$ such that $F(\underline{n}) = 1$. So $s \circ p_{\underline{n}}$ is squash free.

Finally we get the diagram below:



Since B' is an hSet, we can contract the loop $\overline{s \circ p_{\underline{n}}}$ to a point keeping the base point constant.
Hence we can do the same for `loop`, which is provably false in homotopy type theory.

□

By combining the technique before with other ideas, we can also get the following theorems:

Theorem

The following are false in cubical sets, assuming Brouwer's principle. They are not provable in homotopy type theory.

1. $\prod_{\mathbb{N}} \mathbb{S}^1$ *is covered by an hset* $0\text{-}\mathbf{Cov}(\prod_{\mathbb{N}} \mathbb{S}^1)$.
2. *An Escardó-Knapp variant of fullness,* $\mathbf{Full}(\mathbb{N}, 2)_{\mathbf{EK}}$
3. *An Escardó-Knapp variant of collection,* $\mathbf{Coll}_{\mathbf{EK}}$

NB: For proof theoretic reasons it is not necessary to assume Brouwer's principle to show they are not provable in HoTT.

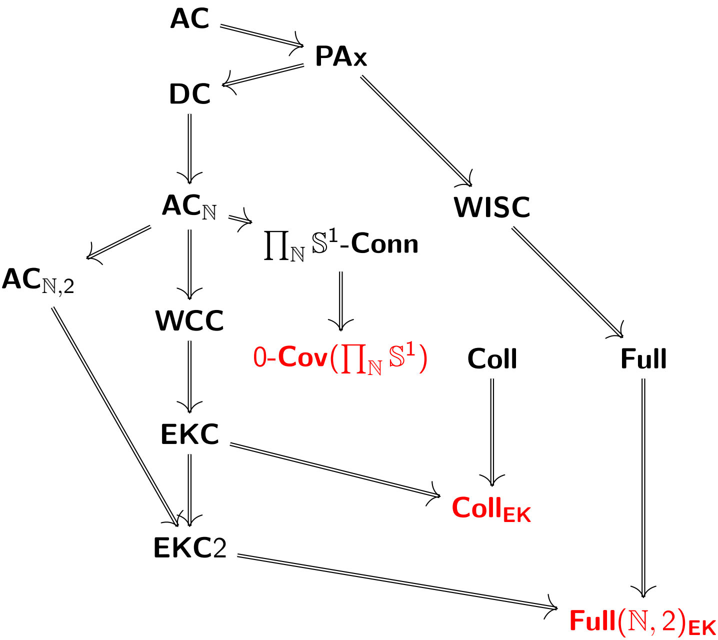
Corollary

The following are false in cubical sets, assuming Brouwer's principle. They are not provable in homotopy type theory.

1. **PAx**
2. *Dependent choice,* **DC**
3. **WISC**
4. *(Type theoretic) Fullness,* **Full**
5. *(Type theoretic) Collection,* **Coll**
6. $\prod_{\mathbb{N}} \mathbb{S}^1$ *is connected,* $\prod_{\mathbb{N}} \mathbb{S}^1$-**Conn**
7. *(Bridges-Richman-Schuster) Weak countable choice,* **WCC**
8. $\mathbf{AC}_{\mathbb{N},2}$
9. *Escardó-Knapp choice,* **EKC**

Proof.

See next slide. □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Corollary

Work over $\mathbf{CZF}_{\mathbf{Exp},\mathbf{Rep}}$*, the theory obtained by replacing subset collection with exponentiation and strong collection with replacement in* **CZF***. The following are not provable.*

1. **PAx**
2. *Dependent choice,* **DC**
3. **WISC**
4. *Fullness,* **Full**
5. *Collection,* **Coll**
6. *(Bridges-Richman-Schuster) Weak countable choice,* **WCC**
7. $\mathbf{AC}_{\mathbb{N},2}$
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Proof.

The HIT cumulative hierarchy models $\mathbf{CZF}_{\mathbf{Exp},\mathbf{Rep}}$ $\mathbf{Full}(\mathbb{N}, 2)_{\mathbf{EK}}$ is "absolute" for the HIT cumulative hierarchy, and $\mathbf{Coll}_{\mathbf{EK}}$ is chosen so that the set theoretic version also fails. □

Open problems:

1. Is there a constructive model of homotopy type theory with countable choice?
2. Are there any other applications of homotopy type theory to constructive set theory? What about classical set theory?

And more philosophically: Is countable choice a reasonable axiom for constructive mathematics?

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Thank you for your attention!