∞-type theories and internal language conjectures

Taichi Uemura   Hoang Kim Nguyen

HoTTEST, 2021/12/02

Contents

- A higher-dimensional generalization of type theories called ∞-type theories.
- A unified formulation of internal language conjectures.
- A proof of Kapulkin and Lumsdaine's internal language conjecture for finitely complete ∞-categories.

Internal language conjecture

Conjecture

Dependent type theory with intensional identity types, dependent function types, univalent universes, and higher inductive types gives internal languages for "elementary ∞-toposes".

Internal language conjecture

Conjecture

Dependent type theory with intensional identity types, dependent function types, univalent universes, and higher inductive types gives internal languages for "elementary ∞-toposes".

The simplest variant:

Conjecture

Dependent type theory with intensional identity types gives internal languages for finitely complete ∞-categories.

Internal language conjecture

Theorem (Kapulkin and Lumsdaine 2018)

There is a canonical functor $\mathrm{H} : \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Lex}_\infty$ *where*

- $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ *is a category of models of* $\mathbb{I}$*, the dependent type theory with intensional identity types;*
- $\mathbf{Lex}_\infty$ *is the* ∞*-category of small* ∞*-categories with finite limits.*

Conjecture (Kapulkin and Lumsdaine 2018)

The functor H *induces an equivalence of* ∞*-categories*

$$\mathrm{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})) \simeq \mathbf{Lex}_\infty$$

where $\mathrm{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}))$ *is a localization, i.e. an* ∞*-category obtained from* $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ *by adjoining formal inverses of certain morphisms.*

Approaches to the internal language conjecture

The functor $\mathrm{H} : \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Lex}_{\infty}$ is decomposed as

$$\begin{tikzcd}
\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \arrow[r] \arrow[rr, bend left, "(1)"] & \mathbf{Tribe} \arrow[r, "\simeq", "(3)"'] & \mathbf{FibCat} \arrow[d, "\simeq"', "(2)"] & \text{1-categories} \\
 & & \mathbf{Lex}_{\infty} & \infty\text{-categories}
\end{tikzcd}$$

($\simeq$ means an equivalence between localizations).

(1) Avigad, Kapulkin, and Lumsdaine (2015) and Gambino and Garner (2008)

(2) Szumiło (2014)

(3) Kapulkin and Szumiło (2019)

Approaches to the internal language conjecture

The functor $\mathrm{H} : \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Lex}_\infty$ is decomposed as

$$\begin{tikzcd}
\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \arrow[r] \arrow[rr, bend left, "(1)"] \arrow[d, dotted, "(4)"'] & \mathbf{Tribe} \arrow[r, "\simeq", "(3)"'] & \mathbf{FibCat} \arrow[d, "\simeq"', "(2)"] & \text{1-categories} \\
\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \arrow[r, dotted, "(4)"'] & \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \arrow[r, dotted, "(4)"'] & \mathbf{Lex}_\infty & \infty\text{-categories}
\end{tikzcd}$$

($\simeq$ means an equivalence between localizations).

(1) Avigad, Kapulkin, and Lumsdaine (2015) and Gambino and Garner (2008)

(2) Szumiło (2014)

(3) Kapulkin and Szumiło (2019)

(4) Our approach: working more ∞-categorically

Problems with 1-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Tribe} \longrightarrow \mathbf{FibCat} \longrightarrow \mathbf{Lex}_{\infty}$$

Problem

Intermediate 1-categories $\mathbf{Tribe}$ *and* $\mathbf{FibCat}$ *are not easy to work with.*

Problems with 1-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Tribe} \longrightarrow \mathbf{FibCat} \longrightarrow \mathbf{Lex}_\infty$$

Problem

Intermediate 1-categories $\mathbf{Tribe}$ *and* $\mathbf{FibCat}$ *are not easy to work with.*

- In $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$, homotopy colimits are easy to compute.
- In $\mathbf{FibCat}$, homotopy limits are easy to compute, but homotopy colimits are not.
- In $\mathbf{Tribe}$, neither homotopy limits nor homotopy colimits are easy to compute.

Problems with 1-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Tribe} \longrightarrow \mathbf{FibCat} \longrightarrow \mathbf{Lex}_\infty$$

Problem

Intermediate 1-categories $\mathbf{Tribe}$ *and* $\mathbf{FibCat}$ *are not easy to work with.*

- In $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$, homotopy colimits are easy to compute.
- In $\mathbf{FibCat}$, homotopy limits are easy to compute, but homotopy colimits are not.
- In $\mathbf{Tribe}$, neither homotopy limits nor homotopy colimits are easy to compute.
- How to generalize?

Problems with 1-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Tribe} \longrightarrow \mathbf{FibCat} \longrightarrow \mathbf{Lex}_{\infty}$$

Problem

Intermediate 1-categories $\mathbf{Tribe}$ *and* $\mathbf{FibCat}$ *are not easy to work with.*

- In $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$, homotopy colimits are easy to compute.
- In $\mathbf{FibCat}$, homotopy limits are easy to compute, but homotopy colimits are not.
- In $\mathbf{Tribe}$, neither homotopy limits nor homotopy colimits are easy to compute.
- How to generalize?
- The coherence problem is not solved at once: the equivalence $\mathrm{L}(\mathbf{FibCat}) \simeq \mathbf{Lex}_{\infty}$ is a kind of strictification, but pullbacks in $\mathcal{C} \in \mathbf{FibCat}$ are still up to isomorphism.

∞-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \longrightarrow \mathbf{Lex}_\infty$$

- $\mathbb{I}_\infty$ and $\mathbb{E}_\infty$ are ∞-type theories.
- $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{T})$'s are *presentable* ∞-categories, so they have limits and colimits, and adjoint functor theorems are available.

∞-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \longrightarrow \mathbf{Lex}_\infty$$

- $\mathbb{I}_\infty$ and $\mathbb{E}_\infty$ are ∞-type theories.
- $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{T})$'s are *presentable* ∞-categories, so they have limits and colimits, and adjoint functor theorems are available.
- All but the last step are formulated within the language of ∞-type theories.
- Easy to generalize.

∞-categorical approach

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \longrightarrow \mathbf{Lex}_\infty$$

- $\mathbb{I}_\infty$ and $\mathbb{E}_\infty$ are ∞-type theories.
- $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{T})$'s are *presentable* ∞-categories, so they have limits and colimits, and adjoint functor theorems are available.
- All but the last step are formulated within the language of ∞-type theories.
- Easy to generalize.
- The coherence problem arises only at the first step $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$.

Goal

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \longrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \longrightarrow \mathbf{Lex}_\infty$$

Theorem

(1) *The composite* $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Lex}_\infty$ *coincides with the functor considered by Kapulkin and Lumsdaine.*

(2) *It induces an equivalence* $\mathbf{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})) \simeq \mathbf{Lex}_\infty$.

Outline

Outline

∞-type theories

Idea

∞-type theories are a higher dimensional generalization of type theories.

∞-type theories

Idea

∞-type theories are a higher dimensional generalization of type theories.

- *Informally*, type theories with *proof-relevant* judgmental equality.

Type theory	∞-type theory
$A_1 \equiv A_2$	$p : A_1 \equiv A_2$
$a_1 \equiv a_2 : A$	$p : a_1 \equiv a_2 : A$
	$q : p_1 \equiv p_2 : a_1 \equiv a_2 : A$

Cf. *explicit conversion* (Curien 1993; Geuvers and Wiedijk 2008).

∞-type theories

Idea

∞-type theories are a higher dimensional generalization of type theories.

- *Informally*, type theories with *proof-relevant* judgmental equality.

Type theory	∞-type theory
$A_1 \equiv A_2$	$p : A_1 \equiv A_2$
$a_1 \equiv a_2 : A$	$p : a_1 \equiv a_2 : A$
	$q : p_1 \equiv p_2 : a_1 \equiv a_2 : A$

  Cf. *explicit conversion* (Curien 1993; Geuvers and Wiedijk 2008).
- Formally, an ∞-categorical generalization of categories with representable maps (Uemura 2019).

Categories with representable maps

Definition

A morphism $u : x \to y$ in a category $\mathcal{C}$ with finite limits is *exponentiable* if the pullback functor $u^* : \mathcal{C}/y \to \mathcal{C}/x$ has a right adjoint u_* called the *pushforward*.

Categories with representable maps

Definition

A morphism $u : x \to y$ in a category $\mathcal{C}$ with finite limits is *exponentiable* if the pullback functor $u^* : \mathcal{C}/y \to \mathcal{C}/x$ has a right adjoint u_* called the *pushforward*.

Definition

A *category with representable maps (CwR)* consists of:

- a category $\mathcal{C}$ with finite limits;
- a class R of exponentiable morphisms in $\mathcal{C}$ satisfying some stability conditions.

Morphisms in R are called *representable maps*.

Categories with representable maps

Example

For any category $\mathcal{C}$, the presheaf category $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ is a CwR where $f : B \to A$ is representable if for any $x \in \mathcal{C}$ and any $a : よ x \to A$, the pullback $a^* B$ is representable.

$$\begin{tikzcd} よ(x ._f a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ よ x \arrow[r, "a"'] & A \end{tikzcd}$$

The representing object $x ._f a \in \mathcal{C}$ is called the *context extension along* f.

$よ : \mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ is the Yoneda embedding.

Type theories

Definition

A *type theory* is a (small) CwR.

- A type theory is an *essentially algebraic theory*.
- Pushforwards along representable maps are used for expressing *variable binding* (cf. logical frameworks (Harper, Honsell, and Plotkin 1993)).

Type theories

Definition

A *type theory* is a (small) CwR.

- A type theory is an *essentially algebraic theory*.
- Pushforwards along representable maps are used for expressing *variable binding* (cf. logical frameworks (Harper, Honsell, and Plotkin 1993)).

Definition

A *model* of a type theory T consists of:

- a category $M(\star)$ with a final object $\diamond$;
- a morphism of CwRs $M : T \to \mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Set})$.

Models of T form a category $\mathbf{Mod}(T)$.

Taichi Uemura, Hoang Kim Nguyen

Type theories

Example

Let $\mathbb{D}$ be the type theory *presented by*

- objects $\mathbf{U}$ and $\mathbf{E}$;
- a representable map $\partial : \mathbf{E} \to \mathbf{U}$.

Type theories

Example

Let $\mathbb{D}$ be the type theory *presented by*

- objects U and E;
- a representable map $\partial : \mathrm{E} \to \mathrm{U}$.

A model of $\mathbb{D}$ consists of:

- a category $M(\star)$ with a final object $\diamond$;
- a representable map $M(\partial) : M(\mathrm{E}) \to M(\mathrm{U})$ of presheaves over $M(\star)$.

This is nothing but a *natural model* (Awodey 2018; Fiore 2012), equivalently a *category with families* (Dybjer 1996).

Intensional type theory

Example

Let $\mathbb{I}$ be the extension of $\mathbb{D}$ by

- a commutative square

$$\begin{tikzcd} E \arrow[r, dotted, "\mathrm{refl}"] \arrow[d, "\Delta"'] & E \arrow[d, "\partial"] \\ E \times_U E \arrow[r, dotted, "\mathrm{Id}"'] & U; \end{tikzcd}$$

- a *path induction operator* (defined as a morphism and an equation);
- (1 and Σ).

Contextual models

Definition

Let M be a model of a type theory T.

(1) The class of *contextual objects* in $M(\star)$ is inductively defined:
   - final objects of $M(\star)$ are contextual;
   - for any $\Gamma \in M(\star)$, $u : y \to x$ a representable map in T, and $A : よ\Gamma \to M(x)$, if Γ is contextual so is $\Gamma ._{u} A$.

(2) M is *contextual* if every object of $M(\star)$ is contextual.

$\mathbf{Mod}^{\mathrm{ctx}}(T) \subset \mathbf{Mod}(T)$ the full subcategory of contextual models.

Contextual models

Definition

Let M be a model of a type theory T.

(1) The class of *contextual objects* in $M(\star)$ is inductively defined:

- final objects of $M(\star)$ are contextual;
- for any $\Gamma \in M(\star)$, $u : y \to x$ a representable map in T, and $A : \text{よ}\,\Gamma \to M(x)$, if Γ is contextual so is $\Gamma ._{u} A$.

(2) M is *contextual* if every object of $M(\star)$ is contextual.

$\mathbf{Mod}^{\mathrm{ctx}}(T) \subset \mathbf{Mod}(T)$ the full subcategory of contextual models.

Example

$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{D})$ is equivalent to the category of contextual categories (Cartmell 1978) (and thus to the category of generalized algebraic theories).

Contextual models

Theorem

For any type theory $\mathbb{T}$, *we have an equivalence*

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{T}) \simeq \mathbf{Lex}(\mathbb{T}, \mathbf{Set})$$

that sends $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{T})$ *to the functor*

$$\mathbb{T} \xrightarrow{M} \mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Set}) \xrightarrow{\mathrm{ev}_\diamond} \mathbf{Set}$$

Contextual models

Corollary

For any morphism $F : T \to S$ *of type theories, we have an adjunction*

$$\begin{tikzcd} \mathbf{Mod}^{\mathrm{ctx}}(T) \arrow[rr, bend left, "F_!"] \arrow[rr, phantom, "\perp"] & & \mathbf{Mod}^{\mathrm{ctx}}(S) \arrow[ll, bend left, "F^*"] \end{tikzcd}$$

Remark

We also have $F^* : \mathbf{Mod}(S) \to \mathbf{Mod}(T)$, but it need not coincide with $F^* : \mathbf{Mod}^{\mathrm{ctx}}(S) \to \mathbf{Mod}^{\mathrm{ctx}}(T)$ unless it preserves contextual models.

∞-type theories

Everything makes sense in the ∞-categorical context.

Definition

An ∞-*type theory* is an ∞-CwR. An n-*type theory* is an ∞-type theory whose underlying ∞-category is an n-category.

Definition

A *model* of an ∞-type theory T consists of:

- an ∞-category $M(\star)$ with a final object;
- a morphism $M : T \to \mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$ of ∞-CwRs.

Theorem

$\mathbf{Mod}^{\mathrm{ctx}}(T) \simeq \mathbf{Lex}(T, \mathbf{Space})$.

Outline

General internal language conjecture

Let T be a 1-type theory.

(1) Define an analogous ∞-type theory T_∞ so T is the 1-*truncation* of T_∞.

(2) Define an ∞-type theory $\mathrm{T}^{\mathrm{ex}}_\infty$ by adding to T some *extensionality* axioms.

(3) We have a span $\mathrm{T} \xleftarrow{\tau} \mathrm{T}_\infty \xrightarrow{\gamma} \mathrm{T}^{\mathrm{ex}}_\infty$ which induces

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}) \xrightarrow{\tau^*} \mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}_\infty) \xrightarrow{\gamma_!} \mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}^{\mathrm{ex}}_\infty).$$

General internal language conjecture

Let T be a 1-type theory.

(1) Define an analogous ∞-type theory T_∞ so T is the 1-*truncation* of T_∞.

(2) Define an ∞-type theory $\mathrm{T}_\infty^{\mathrm{ex}}$ by adding to T some *extensionality* axioms.

(3) We have a span $\mathrm{T} \xleftarrow{\tau} \mathrm{T}_\infty \xrightarrow{\gamma} \mathrm{T}_\infty^{\mathrm{ex}}$ which induces

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}) \xrightarrow{\tau^*} \mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}_\infty) \xrightarrow{\gamma_!} \mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}_\infty^{\mathrm{ex}}).$$

Task

(1) Find a concrete ∞-category $\mathcal{X} \to \mathbf{Cat}_\infty$ equivalent to $\mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}_\infty^{\mathrm{ex}})$.

(2) Prove $\mathbf{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T})) \simeq \mathbf{Mod}^{\mathrm{ctx}}(\mathrm{T}_\infty^{\mathrm{ex}})$.

Intensional ∞-type theory

Example

Let $\mathbb{I}_\infty$ be the ∞-type theory presented by the same data as $\mathbb{I}$, i.e.

- a representable map $\partial : E \to U$;
- a homotopy commutative square

$$
\begin{tikzcd}
E \arrow[r, dotted, "\mathsf{refl}"] \arrow[d, "\Delta"'] & E \arrow[d, "\partial"] \\
E \times_U E \arrow[r, dotted, "\mathsf{Id}"'] & U
\end{tikzcd}
$$

  (the homotopy filling the square is part of data);
- a path induction operator (a morphism and a homotopy for the computation rule);
- (1 and Σ).

Intensional ∞-type theory

Proposition

$\mathbb{I}$ is the 1-truncation of $\mathbb{I}_\infty$: it is the initial 1-type theory equipped a morphism $\tau : \mathbb{I}_\infty \to \mathbb{I}$.

Proposition

$\tau^* : \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ is fully faithful, and its essential image is those $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ with $M(U)$ and $M(E)$ 0-truncated presheaves.

Extensional ∞-type theory

Example

Let $\mathbb{E}_\infty$ be the ∞-type theory obtained from $\mathbb{I}_\infty$ as follows:

- make the identity types *extensional*: make the square

$$\begin{tikzcd} E \arrow[r, "\mathsf{refl}"] \arrow[d, "\Delta"'] & E \arrow[d, "\partial"] \\ E \times_U E \arrow[r, "\mathsf{Id}"'] & U \end{tikzcd}$$

  a pullback (or invert the induced morphism $E \to \mathsf{Id}^* E$);
- make $\partial : E \to U$ *univalent*: (next few slides).

$\mathbb{E}_\infty$ is equipped with a morphism $\gamma : \mathbb{I}_\infty \to \mathbb{E}_\infty$.

Univalent representable maps

Recall the definition of *univalent maps* in ∞-categories (Gepner and Kock 2017; Rasekh 2018, 2021).

Proposition

Let $u : y \to x$ be a representable map in an ∞-CwR $\mathcal{C}$. One can construct an object $\underline{\mathrm{Equiv}}(u) \in \mathcal{C}/x \times x$ classifying equivalences between fibers of u.

Proof.

Because u is exponentiable. □

Precisely, for any object $(v_1, v_2) : z \to x \times x$ of $\mathcal{C}/x \times x$, the mapping space $\mathcal{C}/x \times x(z, \underline{\mathrm{Equiv}}(u))$ is naturally equivalent to the space of equivalences $v_1^* y \simeq v_2^* y$ over z.

Univalent representable maps

We have the section $|\mathrm{id}|_{\mathfrak{u}} : x \to \underline{\mathrm{Equiv}}(\mathfrak{u})$ over $\Delta : x \to x \times x$ corresponding to the identity $\mathfrak{y} \simeq \mathfrak{y}$.

Definition

$\mathfrak{u}$ is *univalent* if the morphism $|\mathrm{id}|_{\mathfrak{u}} : x \to \underline{\mathrm{Equiv}}(\mathfrak{u})$ is an equivalence.

Univalent representable maps

We have the section $|\mathrm{id}|_u : x \to \underline{\mathrm{Equiv}}(u)$ over $\Delta : x \to x \times x$ corresponding to the identity $y \simeq y$.

Definition

u is *univalent* if the morphism $|\mathrm{id}|_u : x \to \underline{\mathrm{Equiv}}(u)$ is an equivalence.

Example

When $\mathcal{C}$ has a *generic representable map*, i.e. any representable map is a pullback of the generic one in a *unique* way, the generic representable map is univalent. For example, $\mathbf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathbf{Space})$ for any $\mathcal{D}$ has one (because the class of representable maps is a bounded local class).

Extensional ∞-type theory

Example

Let $\mathbb{E}_\infty$ be the ∞-type theory obtained from $\mathbb{I}_\infty$ as follows:

- make the identity types *extensional*: make the square

$$\begin{tikzcd} E \arrow[r, "\mathsf{refl}"] \arrow[d, "\Delta"'] & E \arrow[d, "\partial"] \\ E \times_U E \arrow[r, "\mathsf{Id}"'] & U \end{tikzcd}$$

  a pullback (or invert the induced morphism $E \to \mathsf{Id}^* E$);
- make $\partial : E \to U$ *univalent*: invert the morphism $|\mathrm{id}|_\partial : U \to \underline{\mathrm{Equiv}}(\partial)$.

$\mathbb{E}_\infty$ is equipped with a morphism $\gamma : \mathbb{I}_\infty \to \mathbb{E}_\infty$.

(cf. Bocquet 2021, HoTTEST talk)

Internal (∞-)languages for finitely complete ∞-categories

∞-analogue of (Clairambault and Dybjer 2011, 2014).

Theorem

The forgetful functor $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \ni M \mapsto M(\star) \in \mathbf{Cat}_\infty$ *factors through* $\mathbf{Lex}_\infty \subset \mathbf{Cat}_\infty$ *and induces an equivalence*

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \simeq \mathbf{Lex}_\infty.$$

Proof.

The inverse functor maps a $\mathcal{C} \in \mathbf{Lex}_\infty$ to the generic representable map in $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Space})$. $\square$

Family of internal language conjectures

- Base case:

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty) \simeq \mathbf{Lex}_\infty$$

- With Π-types (and function extensionality):

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}^{\Pi}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}^{\Pi}_\infty) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}^{\Pi}_\infty) \simeq \mathbf{LCCC}_\infty$$

- With Π-types and natural numbers:

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}^{\Pi,\mathsf{Nat}}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}^{\Pi,\mathsf{Nat}}_\infty) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}^{\Pi,\mathsf{Nat}}_\infty) \simeq \mathbf{LCCC}^{\mathsf{Nat}}_\infty$$

Family of internal language conjectures

- With Π-types and a countable chain of univalent universes:

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}^{\Pi,\mathcal{U}_{<\omega}}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty^{\Pi,\mathcal{U}_{<\omega}}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty^{\Pi,\mathcal{U}_{<\omega}}) \simeq \mathbf{LCCC}_\infty^{\mathcal{U}_{<\omega}}$$

  where a $\mathcal{C} \in \mathbf{LCCC}_\infty^{\mathcal{U}_{<\omega}}$ has a countable chain of univalent universes *as part of structure*.

- With Π-types and S^1:

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}^{\Pi,\mathsf{S}^1}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty^{\Pi,\mathsf{S}^1}) \to \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty^{\Pi,\mathsf{S}^1}) \simeq \mathbf{LCCC}_\infty^{\mathsf{S}^1}$$

Internal languages for finitely complete ∞-categories

Theorem

The composite

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \xrightarrow{\tau^*} \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \xrightarrow{\gamma_!} \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty)$$

induces an equivalence $\mathbf{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})) \simeq \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty)$. *Consequently, we have*

$$\mathbf{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})) \simeq \mathbf{Lex}_\infty.$$

Moreover, the functor $\gamma_!\tau^* : \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \to \mathbf{Lex}_\infty$ *coincides with the one considered by Kapulkin and Lumsdaine.*

Internal languages for finitely complete ∞-categories

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \xrightarrow{\tau^*} \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \xrightarrow{\gamma_!} \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty)$$

Idea

Once we prove that $\gamma_!\tau^*$ preserves homotopy colimits, the rest is not hard.

Internal languages for finitely complete ∞-categories

$$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \xrightarrow{\tau^*} \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty) \xrightarrow{\gamma_!} \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{E}_\infty)$$

Idea

Once we prove that $\gamma_!\tau^*$ preserves homotopy colimits, the rest is not hard.

(1) Kapulkin and Lumsdaine (2018) showed that $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ is equipped with a structure of a *cofibration category*. In particular, certain colimits in $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ are homotopy colimits.

(2) Prove that τ^* preserves those colimits (and thus so does $\gamma_!\tau^*$).

(3) Then, it suffices to check a couple of conditions called the *left approximation property* (Cisinski 2019).

(4) The last assertion is proved by checking that both have the same universal property.

Cofibrations in $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$

Theorem (Kapulkin and Lumsdaine 2018)

$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ *is equipped with a structure of a cofibration category (as part of a combinatorial left semi-model structure).*

Definition

A *cofibration* in $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ is a retract of an extension by types and terms but no equation. An $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ is *cofibrant* if $0 \to M$ is a cofibration.

Cofibrations in $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$

Theorem (Kapulkin and Lumsdaine 2018)

$\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ is equipped with a structure of a cofibration category (as part of a combinatorial left semi-model structure).

Definition

A *cofibration* in $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ is a retract of an extension by types and terms but no equation. An $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ is *cofibrant* if $0 \to M$ is a cofibration.

Definition

One can define the cofibrations in $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ in the same way as $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$.

Coherence theorem

The hardest part in our proof.

Theorem

Any cofibrant object of $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ *belongs to* $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \subset \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$.

- That is, in a "free" model of $\mathbb{I}_\infty$, every diagram of homotopies commutes.
- This is the only place where the coherence problem comes in.

Corollary

$\tau^* : \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \hookrightarrow \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ *preserves initial objects and pushouts of cofibrations along arbitrary morphisms between cofibrant objects.*

Outline

Approaches to the coherence problem

Theorem

Any cofibrant object of $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ *belongs to* $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \subset \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$.

Split replacement: For any $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$, find a $\mathrm{Spl}\, M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ and a trivial fibration $\mathrm{Spl}\, M \to M$ (cf. Hofmann 1995). In particular, if M is cofibrant, it is a retract of $\mathrm{Spl}\, M$.

Approaches to the coherence problem

Theorem

Any cofibrant object of $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ *belongs to* $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}) \subset \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$.

Split replacement For any $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$, find a $\mathrm{Spl}\, M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I})$ and a trivial fibration $\mathrm{Spl}\, M \to M$ (cf. Hofmann 1995). In particular, if M is cofibrant, it is a retract of $\mathrm{Spl}\, M$.

Rewriting (cf. Curien 1993; Mac Lane 1963).

Normalization (by evaluation) Expect

$$\text{Normalizing} \implies \text{Decidable equality} \implies 0\text{-truncated}$$

Split replacement

An $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ consists of

- an ∞-category $M(\star)$ with a final object;
- a representable map $M(\partial) : M(E) \to M(U)$ in $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$;
- an Id-type structure.

Idea

(1) *Present the ∞-topos $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$ by a model category $\mathcal{X}$.*

Split replacement

An $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ consists of

- an ∞-category $M(\star)$ with a final object;
- a representable map $M(\partial) : M(E) \to M(U)$ in $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$;
- an Id-type structure.

Idea

(1) *Present the ∞-topos $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$ by a model category $\mathcal{X}$.*

(2) $M(\partial)$ *is represented by a universe $\partial_{\mathcal{X}} : E_{\mathcal{X}} \to U_{\mathcal{X}}$ in $\mathcal{X}$.*

Split replacement

An $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ consists of

- an ∞-category $M(\star)$ with a final object;
- a representable map $M(\partial) : M(E) \to M(U)$ in $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$;
- an Id-type structure.

Idea

(1) *Present the ∞-topos $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$ by a model category $\mathfrak{X}$.*

(2) $M(\partial)$ *is represented by a universe $\partial_{\mathfrak{X}} : E_{\mathfrak{X}} \to U_{\mathfrak{X}}$ in $\mathfrak{X}$.*

(3) *Use Voevodsky's universe method to obtain a contextual natural model* $\operatorname{Spl} M$ *from $\partial_{\mathfrak{X}}$.*

Split replacement

An $M \in \mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$ consists of

- an ∞-category $M(\star)$ with a final object;
- a representable map $M(\partial) : M(E) \to M(U)$ in $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$;
- an Id-type structure.

Idea

(1) *Present the ∞-topos $\mathbf{Fun}(M(\star)^{\mathrm{op}}, \mathbf{Space})$ by a model category $\mathcal{X}$.*

(2) $M(\partial)$ *is represented by a universe $\partial_{\mathcal{X}} : E_{\mathcal{X}} \to U_{\mathcal{X}}$ in $\mathcal{X}$.*

(3) *Use Voevodsky's universe method to obtain a contextual natural model $\mathrm{Spl}\, M$ from $\partial_{\mathcal{X}}$.*

(4) *Lift the Id-type structure so $\mathrm{Spl}\, M$ is a model of $\mathbb{I}$.*

Type-theoretic model topos

We choose $\mathcal{X}$ to be a *type-theoretic model topos* (Shulman 2019).

- $\mathcal{X}$ is a Grothendieck topos.
- The cofibrations are precisely the monomorphisms.
- Right proper, so the localization functor $\gamma_{\mathcal{X}} : \mathcal{X} \to \mathbf{L}\,\mathcal{X}$ preserves pushforwards of fibrations between fibrant objects.
- Enough univalent universes (not needed for Id, but useful for lifting 1, Σ, and Π).

Voevodsky's universe method

There exists a fibration $\partial_{\mathfrak{X}} : \mathrm{E}_{\mathfrak{X}} \to \mathrm{U}_{\mathfrak{X}}$ between fibrant objects in $\mathfrak{X}$ sent to $\mathrm{M}(\partial)$ by the localization functor $\gamma_{\mathfrak{X}} : \mathfrak{X} \to \mathbf{L}\,\mathfrak{X} \simeq \mathbf{Fun}(\mathrm{M}(\star)^{\mathrm{op}}, \boldsymbol{\mathcal{S}\mathrm{pace}})$. Define a contextual natural model $\mathrm{Spl}\,\mathrm{M}$ as follows:

(1) $(\mathfrak{X}, よ\,\partial_{\mathfrak{X}} : よ\,\mathrm{E}_{\mathfrak{X}} \to よ\,\mathrm{U}_{\mathfrak{X}})$ defines a natural model;

(2) restrict the base category to the full subcategory spanned by the contextual objects.

Voevodsky's universe method

There exists a fibration $\partial_{\mathcal{X}} : \mathrm{E}_{\mathcal{X}} \to \mathrm{U}_{\mathcal{X}}$ between fibrant objects in $\mathcal{X}$ sent to $\mathrm{M}(\partial)$ by the localization functor $\gamma_{\mathcal{X}} : \mathcal{X} \to \mathbf{L}\,\mathcal{X} \simeq \mathbf{Fun}(\mathrm{M}(\star)^{\mathrm{op}}, \boldsymbol{\mathcal{S}pace})$. Define a contextual natural model $\operatorname{Spl} \mathrm{M}$ as follows:

(1) $(\mathcal{X}, よ\, \partial_{\mathcal{X}} : よ\, \mathrm{E}_{\mathcal{X}} \to よ\, \mathrm{U}_{\mathcal{X}})$ defines a natural model;

(2) restrict the base category to the full subcategory spanned by the contextual objects.

Concretely,

- $(\operatorname{Spl} \mathrm{M})(\star) \subset \mathcal{X}$;
- $\Gamma \in (\operatorname{Spl} \mathrm{M})(\star)$ if $\Gamma \to 1$ is a composite of pullbacks of $\partial_{\mathcal{X}}$;
- $(\operatorname{Spl} \mathrm{M})(\mathrm{U})(\Gamma) = \mathcal{X}(\Gamma, \mathrm{U}_{\mathcal{X}})$;
- $(\operatorname{Spl} \mathrm{M})(\mathrm{E})(\Gamma) = \mathcal{X}(\Gamma, \mathrm{E}_{\mathcal{X}})$.

(Cf. Voevodsky 2015)

Lifting Id-type structure

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, "\gamma_{\mathcal{X}}"'] \\
\mathbf{L}\,\mathcal{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{\mathcal{X}} \arrow[dr, "\Delta"'] & & & E_{\mathcal{X}} \arrow[dr, "\partial_{\mathcal{X}}"] & \\
& E_{\mathcal{X}} \times_{U_{\mathcal{X}}} E_{\mathcal{X}} & & & U_{\mathcal{X}}
\end{tikzcd}
$$

$$\mapsto$$

$$
\begin{tikzcd}
M(E) \arrow[rr, "M(\mathrm{refl})"] \arrow[dr, "M(\Delta)"'] & & M(E) \arrow[dr, "M(\partial)"] & \\
& M(E \times_U E) \arrow[rr, "M(\mathrm{Id})"'] & \sim & M(U)
\end{tikzcd}
$$

($U_{\mathcal{X}}$ is fibrant, $\partial_{\mathcal{X}} : E_{\mathcal{X}} \to U_{\mathcal{X}}$ is a fibration, and all objects are cofibrant.)

Lifting Id-type structure

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, "\gamma_{\mathcal{X}}"'] \\
\mathbf{L}\,\mathcal{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{\mathcal{X}} \arrow[dr, "\Delta"'] & & E_{\mathcal{X}} \arrow[dr, "\partial_{\mathcal{X}}"] & \\
& E_{\mathcal{X}} \times_{U_{\mathcal{X}}} E_{\mathcal{X}} \arrow[rr, dotted, "\mathsf{Id}_{\mathcal{X}}"'] & & U_{\mathcal{X}}
\end{tikzcd}
$$

$$\downarrow$$

$$
\begin{tikzcd}
M(E) \arrow[rr, "M(\mathsf{refl})"] \arrow[dr, "M(\Delta)"'] & & M(E) \arrow[dr, "M(\partial)"] & \\
& M(E \times_U E) \arrow[rr, "M(\mathsf{Id})"'] \arrow[ur, phantom, "\sim"] & & M(U)
\end{tikzcd}
$$

($U_{\mathcal{X}}$ is fibrant, $\partial_{\mathcal{X}} : E_{\mathcal{X}} \to U_{\mathcal{X}}$ is a fibration, and all objects are cofibrant.)

Lifting Id-type structure

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\gamma_{\mathcal{X}}"'] \\ \mathbf{L}\,\mathcal{X} \end{tikzcd}$$

$$\begin{tikzcd} E_{\mathcal{X}} \arrow[rr, dotted, "\mathsf{refl}_{\mathcal{X}}"] \arrow[dr, "\Delta"'] & & E_{\mathcal{X}} \arrow[d, "\partial_{\mathcal{X}}"] \\ & E_{\mathcal{X}} \times_{\mathcal{U}_{\mathcal{X}}} E_{\mathcal{X}} \arrow[r, dotted, "\mathsf{Id}_{\mathcal{X}}"'] \arrow[ur, phantom, "="] & \mathcal{U}_{\mathcal{X}} \end{tikzcd}$$

$$\downarrow$$

$$\begin{tikzcd} M(E) \arrow[rr, "M(\mathsf{refl})"] \arrow[dr, "M(\Delta)"'] & & M(E) \arrow[d, "M(\partial)"] \\ & M(E \times_{\mathcal{U}} E) \arrow[r, "M(\mathsf{Id})"'] \arrow[ur, phantom, "\sim"] & M(\mathcal{U}) \end{tikzcd}$$

($\mathcal{U}_{\mathcal{X}}$ is fibrant, $\partial_{\mathcal{X}} : E_{\mathcal{X}} \to \mathcal{U}_{\mathcal{X}}$ is a fibration, and all objects are cofibrant.)

Lifting path induction

$$
\begin{tikzcd}
\mathcal{X} \arrow[dddd, "\gamma_{\mathcal{X}}"'] & A \arrow[rr] \arrow[dr, "\mathsf{refl}_A"'] & & E_{\mathcal{X}} \arrow[dr, "\partial_{\mathcal{X}}"] & \\
& & \mathsf{Id}_A \arrow[rr] & & U_{\mathcal{X}} \\
& & {} \arrow[d, mapsto] & & \\
& M(A) \arrow[rr] \arrow[dr, "M(\mathsf{refl}_A)"'] & {} & M(E) \arrow[dr, "M(\partial)"] \arrow[d, phantom, "\sim"] & \\
\mathbf{L}\,\mathcal{X} & & M(\mathsf{Id}_A) \arrow[ur] \arrow[rr] \arrow[u, phantom, "\sim"] & {} & M(U)
\end{tikzcd}
$$

($\partial_{\mathcal{X}}$ is a fibration and $\mathsf{refl}_A : A \to \mathsf{Id}_A$ is a cofibration.)

Lifting path induction

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, "\gamma_{\mathcal{X}}"'] \\
\mathbf{L}\,\mathcal{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\mathsf{refl}_A"'] & & E_{\mathcal{X}} \arrow[dr, "\partial_{\mathcal{X}}"] \\
& \mathsf{Id}_A \arrow[ur, dotted] \arrow[u, phantom, "\sim"] \arrow[rr] & \arrow[u, phantom, "="] & U_{\mathcal{X}}
\end{tikzcd}
$$

$$\mapsto$$

$$
\begin{tikzcd}
M(A) \arrow[rr] \arrow[dr, "M(\mathsf{refl}_A)"'] & & M(E) \arrow[dr, "M(\partial)"] \\
& M(\mathsf{Id}_A) \arrow[ur] \arrow[u, phantom, "\sim"] \arrow[rr] & \arrow[u, phantom, "\sim"] & M(U)
\end{tikzcd}
$$

($\partial_{\mathcal{X}}$ is a fibration and $\mathsf{refl}_A : A \to \mathsf{Id}_A$ is a cofibration.)

Taichi Uemura, Hoang Kim Nguyen

Lifting path induction

$$\begin{tikzcd}
\mathcal{X} \arrow[dd, "\gamma_{\mathcal{X}}"'] & A \arrow[rr] \arrow[dr, tail, "\mathsf{refl}_A"'] & & E_{\mathcal{X}} \arrow[dr, two heads, "\partial_{\mathcal{X}}"] \arrow[d, phantom, "="] & \\
& & \mathsf{Id}_A \arrow[ur, dotted] \arrow[rr] \arrow[u, phantom, "\sim"] & {} & U_{\mathcal{X}} \\
& & \arrow[d, mapsto] & & \\
\mathbf{L}\,\mathcal{X} & M(A) \arrow[rr] \arrow[dr, "M(\mathsf{refl}_A)"'] & & M(E) \arrow[dr, "M(\partial)"] \arrow[d, phantom, "\sim"] & \\
& & M(\mathsf{Id}_A) \arrow[ur] \arrow[rr] \arrow[u, phantom, "\sim"] & {} & M(U)
\end{tikzcd}$$

($\partial_{\mathcal{X}}$ is a fibration and $\mathsf{refl}_A : A \to \mathsf{Id}_A$ is a cofibration.)

Lifting path induction

$$
\begin{tikzcd}
\mathcal{X} \arrow[d, "\gamma_{\mathcal{X}}"'] \\
\mathbf{L}\,\mathcal{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr, hook, "\mathsf{refl}_A"'] & \arrow[d, phantom, "="] & E_{\mathcal{X}} \arrow[dr, two heads, "\partial_{\mathcal{X}}"] & \arrow[d, phantom, "="] & \\
& \mathsf{Id}_A \arrow[ur, dotted] \arrow[rr] & & U_{\mathcal{X}}
\end{tikzcd}
$$

$$\downarrow$$

$$
\begin{tikzcd}
M(A) \arrow[rr] \arrow[dr, "M(\mathsf{refl}_A)"'] & \arrow[d, phantom, "\sim"] & M(E) \arrow[dr, "M(\partial)"] & \arrow[d, phantom, "\sim"] & \\
& M(\mathsf{Id}_A) \arrow[ur] \arrow[rr] & & M(U)
\end{tikzcd}
$$

($\partial_{\mathcal{X}}$ is a fibration and $\mathsf{refl}_A : A \to \mathsf{Id}_A$ is a cofibration.)

Problem with the split replacement

- There seems to be no general way to lift type constructors with judgmental computation rules.
- It works for Id because the constructor refl is a cofibration (monomorphism) for a trivial reason (factorization of the diagonal map).
- For general inductive types, constructors are not necessarily monomorphisms.

Problem with the split replacement

- There seems to be no general way to lift type constructors with judgmental computation rules.
- It works for Id because the constructor refl is a cofibration (monomorphism) for a trivial reason (factorization of the diagonal map).
- For general inductive types, constructors are not necessarily monomorphisms.
- For 1, Σ, and Π, we can replace $\partial_{\mathcal{X}}$ by a weakly equivalent one closed under these type constructors (with a rise in universe levels for Π).

Problem with the split replacement

- There seems to be no general way to lift type constructors with judgmental computation rules.
- It works for Id because the constructor refl is a cofibration (monomorphism) for a trivial reason (factorization of the diagonal map).
- For general inductive types, constructors are not necessarily monomorphisms.
- For 1, Σ, and Π, we can replace ∂_χ by a weakly equivalent one closed under these type constructors (with a rise in universe levels for Π).
- We expect that the other approach, rewriting or normalization, works for a wide range of type constructors, if it works.

Summary

- A higher-dimensional generalization of type theories called ∞-type theories.
- A unified formulation of internal language conjectures.
- Coherence theorem via split replacement for $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_{\infty})$.

Summary

- A higher-dimensional generalization of type theories called ∞-type theories.
- A unified formulation of internal language conjectures.
- Coherence theorem via split replacement for $\mathbf{Mod}^{\mathrm{ctx}}(\mathbb{I}_\infty)$.

Future work:

- Better split replacement, or coherence via rewriting or normalization.
- "Syntax" for ∞-type theories.
- Other applications, say conservativity (cf. Bocquet 2020)? Morita equivalence (Isaev 2020) between T and T' may be replaced by $\mathbf{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathsf{T})) \simeq \mathbf{Mod}^{\mathrm{ctx}}(\mathsf{T}_\infty) \simeq \mathbf{L}(\mathbf{Mod}^{\mathrm{ctx}}(\mathsf{T}'))$ for a suitable ∞-type theory T_∞.

References I

Jeremy Avigad, Krzysztof Kapulkin, and Peter LeFanu Lumsdaine (2015). "Homotopy limits in type theory". In: *Mathematical Structures in Computer Science* 25.5, pp. 1040–1070. DOI: 10.1017/S0960129514000498.

Steve Awodey (2018). "Natural models of homotopy type theory". In: *Mathematical Structures in Computer Science* 28.2, pp. 241–286. DOI: 10.1017/S0960129516000268.

Rafaël Bocquet (2020). *Coherence of strict equalities in dependent type theories*. arXiv: 2010.14166v1.

— (2021). *Coherence of definitional equality in type theory*. Talk at HoTTEST. URL: https://www.youtube.com/watch?v=WKCxBygnYqE.

J. W. Cartmell (1978). "Generalised algebraic theories and contextual categories". PhD thesis. Oxford University.

Denis-Charles Cisinski (2019). *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press. DOI: 10.1017/9781108588737.

References II

Pierre Clairambault and Peter Dybjer (2011). "The Biequivalence of Locally Cartesian Closed Categories and Martin-Löf Type Theories". In: *Typed Lambda Calculi and Applications*. Ed. by Luke Ong. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 91–106. DOI: 10.1007/978-3-642-21691-6_10.

— (2014). "The biequivalence of locally cartesian closed categories and Martin-Löf type theories". In: *Mathematical Structures in Computer Science* 24.6, e240606. DOI: 10.1017/S0960129513000881.

Pierre-Louis Curien (1993). "Substitution up to Isomorphism". In: *Fundam. Inform.* 19.1/2, pp. 51–85.

Peter Dybjer (1996). "Internal Type Theory". In: *Types for Proofs and Programs: International Workshop, TYPES '95 Torino, Italy, June 5–8, 1995 Selected Papers*. Ed. by Stefano Berardi and Mario Coppo. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 120–134. DOI: 10.1007/3-540-61780-9_66.

Marcelo Fiore (2012). *Discrete Generalised Polynomial Functors*. Talk at ICALP 2012. URL: http://www.cl.cam.ac.uk/~mpf23/talks/ICALP2012.pdf.

References III

Nicola Gambino and Richard Garner (2008). "The identity type weak factorisation system". In: *Theoretical Computer Science* 409.1, pp. 94–109. DOI: 10.1016/j.tcs.2008.08.030.

David Gepner and Joachim Kock (Jan. 2017). "Univalence in locally cartesian closed ∞-categories". In: *Forum Mathematicum* 29.3. DOI: 10.1515/forum-2015-0228.

Herman Geuvers and Freek Wiedijk (2008). "A Logical Framework with Explicit Conversions". In: *Electronic Notes in Theoretical Computer Science* 199, pp. 33–47. DOI: 10.1016/j.entcs.2007.11.011.

Robert Harper, Furio Honsell, and Gordon Plotkin (Jan. 1993). "A Framework for Defining Logics". In: *J. ACM* 40.1, pp. 143–184. DOI: 10.1145/138027.138060.

References IV

Martin Hofmann (1995). "On the interpretation of type theory in locally cartesian closed categories". In: *Computer Science Logic*. Ed. by Leszek Pacholski and Jerzy Tiuryn. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 427–441. DOI: 10.1007/BFb0022273.

Valery Isaev (2020). *Morita equivalences between algebraic dependent type theories*. arXiv: 1804.05045v2.

Krzysztof Kapulkin and Peter LeFanu Lumsdaine (2018). "The homotopy theory of type theories". In: *Adv. Math.* 337, pp. 1–38. DOI: 10.1016/j.aim.2018.08.003.

Krzysztof Kapulkin and Karol Szumiło (2019). "Internal languages of finitely complete $(\infty, 1)$-categories". In: *Selecta Math. (N.S.)* 25.2, Art. 33, 46. DOI: 10.1007/s00029-019-0480-0.

Saunders Mac Lane (1963). "Natural associativity and commutativity". In: *Rice Univ. Stud.* 49.4, pp. 28–46.

Nima Rasekh (2018). *Complete Segal Objects*. arXiv: 1805.03561v1.

References V

Nima Rasekh (2021). *Univalence in Higher Category Theory*. arXiv: 2103.12762v1.

Michael Shulman (2019). *All* $(\infty, 1)$*-toposes have strict univalent universes*. arXiv: 1904.07004v2.

Karol Szumiło (2014). "Two Models for the Homotopy Theory of Cocomplete Homotopy Theories". PhD thesis. University of Bonn. arXiv: 1411.0303v1.

Taichi Uemura (2019). *A General Framework for the Semantics of Type Theory*. arXiv: 1904.04097v2.

Vladimir Voevodsky (2015). "A C-system defined by a universe category". In: *Theory and Applications of Categories* 30.37, pp. 1181–1214.