A Univalent Formalization of Affine Schemes in `Cubical Agda`

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Set-level (constructive) mathematics in `Cubical Agda`

- Equality by paths ($x \equiv_A y$) as functions

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⇒ Great for univalent formalization of set-level constructive mathematics in the spirit of Voevodsky [2015]

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All use non-constructive "Hartshorne" approach...

$\Rightarrow$ Formalize constructive "lift from basis" approach in `Cubical Agda` (following Coquand et al. [2009] with crucial help from univalence)

Structure sheaf on the Zariski lattice (over R)

classically: compact open sets
$U \subseteq \mathrm{Spec}(R) = \{\mathfrak{p} \text{ prime ideal}\}$

constructively: f.g. ideals $\mathfrak{a}, \mathfrak{b}$
modulo $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$

$$\mathcal{O} : \mathrm{ZL}_R^{op} \to \mathrm{CommRing}$$

$$D(f) \mapsto R[1/f]$$

generators, $f \in R$

classically: $\{\mathfrak{p} \mid f \notin \mathfrak{p}\}$

constructively: equiv. class of $\langle f \rangle$

ring of fractions
of the form r/f^n

Comparison lemma for sheaves on distributive lattices

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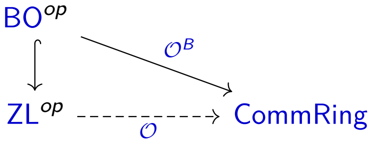
Kan-extension along inclusion of generators (basic opens)
preserves sheaf property



Comparison lemma for sheaves on distributive lattices

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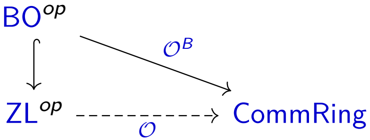

For Kan-extension to exist need *small* ZL!
Construction due to Español [1983]:

$$\mathsf{ZL} = \mathsf{List}\ R\ /\ \_\!\sim\!\_$$

$$\begin{aligned} [x_1, \ldots, x_n] \sim [y_1, \ldots, y_m] \quad &= \quad \forall i.\ x_i \in \sqrt{\langle\ y_1, \ldots, y_m\ \rangle} \\ &\& \quad \forall i.\ y_i \in \sqrt{\langle\ x_1, \ldots, x_n\ \rangle} \end{aligned}$$

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Textbook-argument: $D(f) = D(g) \Rightarrow$ *canonical iso* $R[1/f] \cong R[1/g]$

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$$\Leftrightarrow\; \exists!\, \varphi : R[1/g] \to R[1/f] \text{ s.t. } \varphi(x/1) = x/1 \text{ for } x \in R$$

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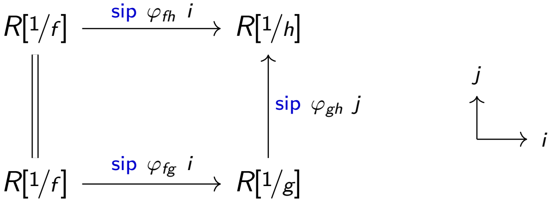
And by using univalence/the SIP for R-Alg:

$$D(f) \equiv D(g) \quad\Rightarrow\quad \mathsf{isContr}\ \big(R[1/f] \equiv R[1/g]\big)$$

(center of contraction: sip applied to unique iso $\varphi_{fg} : R[1/f] \cong R[1/g]$)

Overcoming the h-level mismatch (in R-Alg)

By a result due to Kraus [2015] we need:
for each $f\ g\ h : R$ with $D(f) \equiv D(g) \equiv D(h)$, a filler of the square



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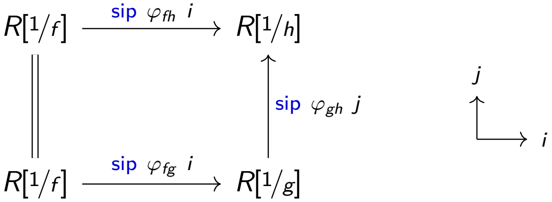


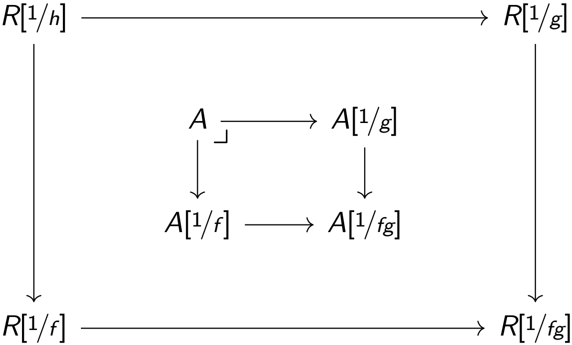
Proof: This is equivalent to giving a path

$$\text{sip } \varphi_{fh} \equiv \text{sip } \varphi_{fg} \bullet \text{sip } \varphi_{gh}$$

Sheaf property for binary cover $D(h) \equiv D(f) \vee D(g)$

Goal: outer square is pullback

Lemma: $\langle f, g \rangle = A \Rightarrow$ pullback square



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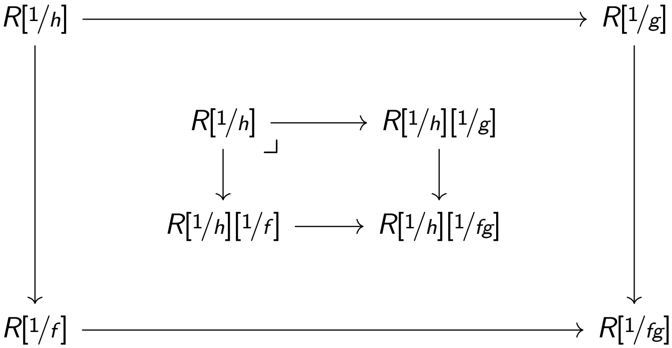
Lemma with $\langle f/1, g/1 \rangle = R[1/h] \Rightarrow$ pullback square



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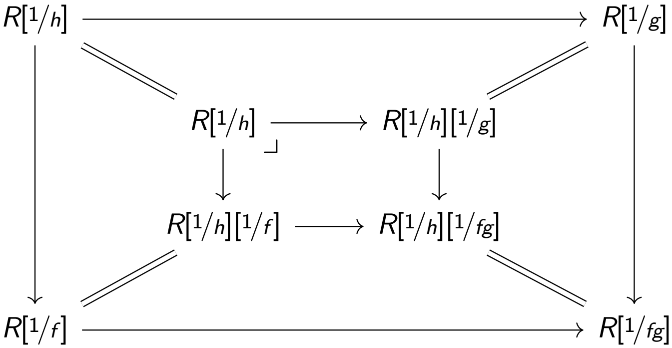

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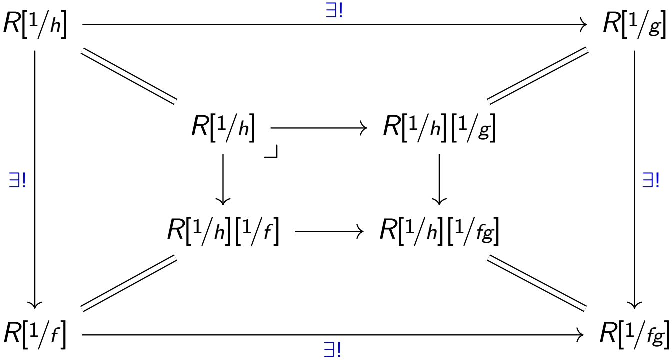

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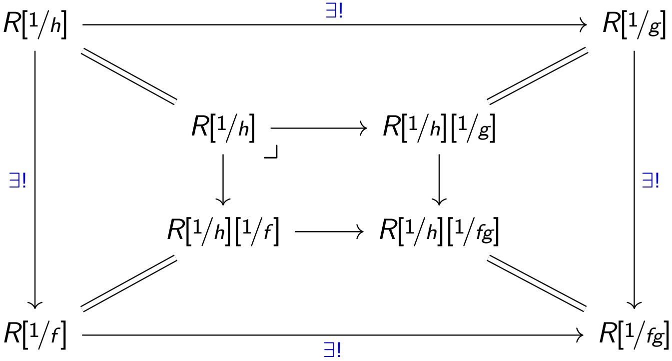
- transport along paths of rings, e.g. $R[1/h][1/f] \equiv R[1/hf] \equiv R[1/f]$
- get dependent paths between morphisms for free
- forgetful functor pres. limits $\Rightarrow$ pullback square in comm. rings



Summary & future work

We presented the outline of a formalization of affine schemes that:

- uses a point-free, constructive Zariski lattice but follows (& elaborates) the textbook strategy $D(f) \mapsto R[1/f]$
- uses a simple algebraic observation and univalence to make the construction work *out of the box*! (sort of)

What lies ahead:

- define (spectral) schemes as ringed lattice
- show that classically those are actually *quasi-compact, quasi-separated schemes*
- projective schemes

Thank You

Sheaves

Idea: restrict sheaf definition for locales to finite covers.

Presheaf $\mathcal{F} : L^{op} \to \mathcal{C}$ is *sheaf on distributive lattice* L iff:

- $\mathcal{F}(\bot)$ is the terminal object in $\mathcal{C}$
- $\forall\, x, y \in L$ the following is a pullback square

$$\begin{array}{ccc}
\mathcal{F}(x \vee y) & \longrightarrow & \mathcal{F}(x) \\
\big\downarrow & \lrcorner & \big\downarrow \\
\mathcal{F}(y) & \longrightarrow & \mathcal{F}(x \wedge y)
\end{array}$$

Links to library

- The Zariski lattice
- General construction of presheaf and lemma for sheaf property
  (lines 532 & 633)
- Key lemma from univalence
  (line 298)

Or just click your way through, starting here (def. of the structure sheaf)

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