Towards Spectral Sequences for Homology

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November 22, 2018

Joint work with Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke and Mike Shulman.

Outline

- Spectra and cohomology
- Spectral sequences for cohomology
- Spectral sequences for homology
- Applications

Spectra

A prespectrum is a sequence of pointed types $Y : \mathbb{Z} \to \mathrm{Type}^*$ with pointed maps $\Sigma Y_n \to^* Y_{n+1}$ called *structure maps*.

By the adjunction $\Sigma \dashv \Omega$, we can equivalently take maps $Y_n \to^* \Omega Y_{n+1}$.

An Ω-spectrum or *spectrum* is a prespectrum Y where the maps $Y_n \to^* \Omega Y_{n+1}$ are equivalences.

A spectrum Y is called n-truncated if Y_k is $(n+k)$-truncated for all $k : \mathbb{Z}$.

The homotopy groups of an Ω-spectrum Y are $\pi_n(Y) :\equiv \pi_{n+k}(Y_k)$ (which is independent of k and also defined for negative n).

Spectra

Examples

- If A is an abelian group, the Eilenberg-MacLane spectrum $HA : \Omega\text{-Spectrum}$ where $(HA)_n = K(A, n)$ is a 0-truncated Ω-spectrum.
- If X and Y are prespectra, then $X \vee Y$ defined by

  $$(X \vee Y)_n :\equiv X_n \vee Y_n$$

  is a prespectrum, since we have a pointed map

  $$\Omega X_{n+1} \vee \Omega Y_{n+1} \to^* \Omega(X_{n+1} \vee Y_{n+1}).$$

- If X is a pointed type and $Y : X \to \Omega\text{-Spectrum}$ is family of spectra parametrized over X we have a spectrum $\Pi^*(x : X),\ Yx$ defined by

  $$(\Pi^*(x : X),\ Yx)_n :\equiv \Pi^*(x : X),\ (Yx)_n$$

Cohomology

If $X : \mathrm{Type}^*$ and $Y : \Omega$-Spectrum, we have generalized reduced cohomology:

$$Y^n(X) \equiv \widetilde{H}^n(X;Y) :\equiv \pi_{-n}(X \to^* Y) \simeq \|X \to^* Y_n\|_0.$$

If $Y = HA$, then we get the ordinary reduced cohomology $\widetilde{H}^n(X;A)$.

If X is any type, we get unreduced cohomology

$$H^n(X;Y) :\equiv \pi_{-n}(X \to Y) \simeq \widetilde{H}^n(X+1;Y).$$

We get parametrized cohomology by replacing functions with dependent functions:

$$\widetilde{H}^n(X;\lambda x.Yx) :\equiv \pi_{-n}(\Pi^*(x:X),\, Yx) \simeq \|\Pi^*(x:X),\, (Yx)_n\|_0.$$

Here $Y : X \to \Omega$-Spectrum is a parametrized spectrum.

Long Exact Sequence of Homotopy Groups

Given a pointed map $f : X \to^* Y$ with fiber F.
Then we have the following long exact sequence.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
\pi_2(F) & \xrightarrow[\pi_2(p_1)]{} & \pi_2(X) & \xrightarrow{\pi_2(f)} & \pi_2(Y) \\
 & \swarrow^{\pi_1(\delta)} & & & \\
\pi_1(F) & \xrightarrow[\pi_1(p_1)]{} & \pi_1(X) & \xrightarrow{\pi_1(f)} & \pi_1(Y) \\
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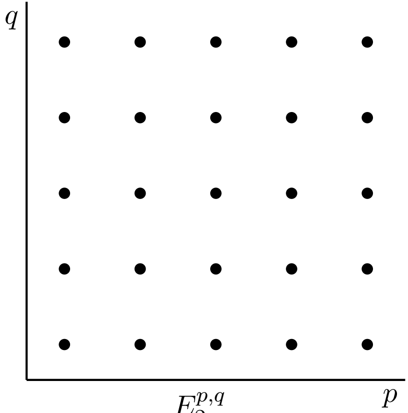
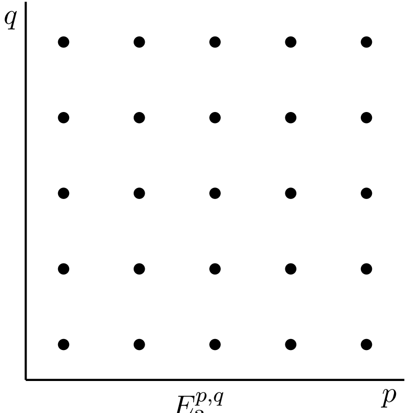
\pi_0(F) & \xrightarrow[\pi_0(p_1)]{} & \pi_0(X) & \xrightarrow{\pi_0(f)} & \pi_0(Y)
\end{array}
$$

Definition. A spectral sequence consists of a family $E_r^{p,q}$ of abelian groups for $p, q : \mathbb{Z}$ and $r \geq 2$. For a fixed r this gives the *r-page* of the spectral sequence. ...

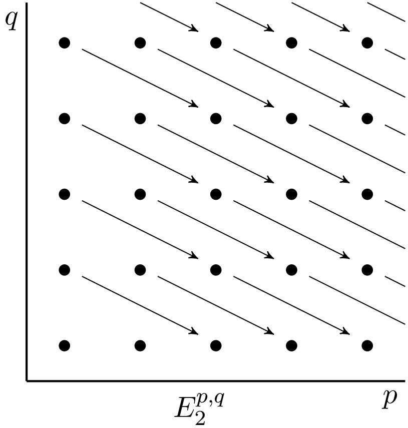


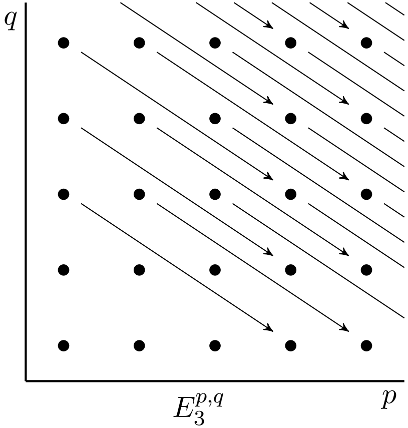
$E_2^{p,q}$



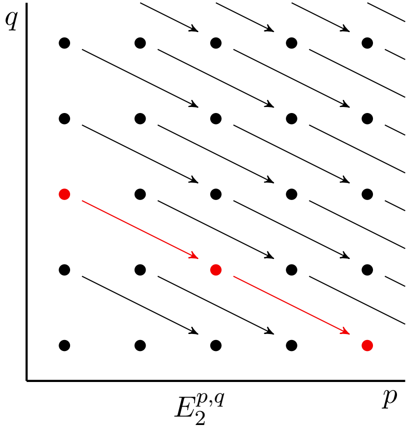
$E_3^{p,q}$

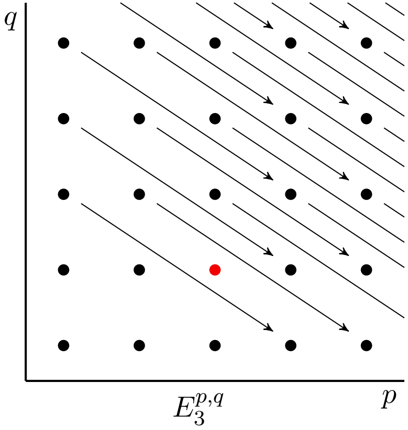
Definition. ... with *differentials* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$ (this is cohomologically indexed) ...





Definition. ... and with isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \simeq E_{r+1}^{p,q}$ where $H^{p,q}(E_r) = \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,q+r-1})$.





Spectral Sequences

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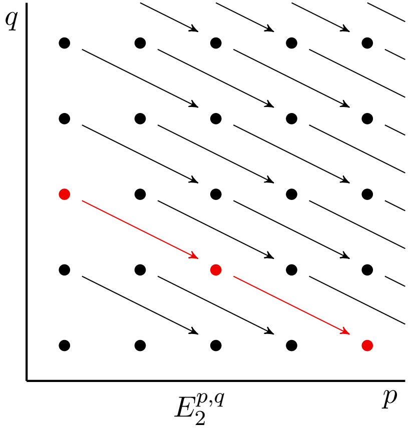
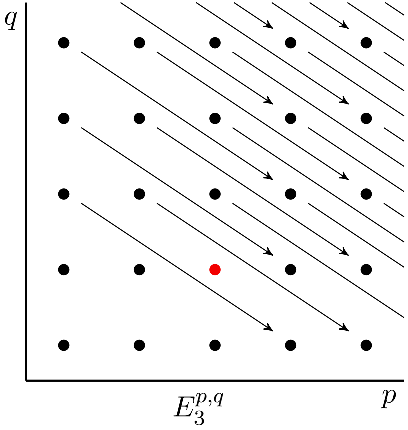
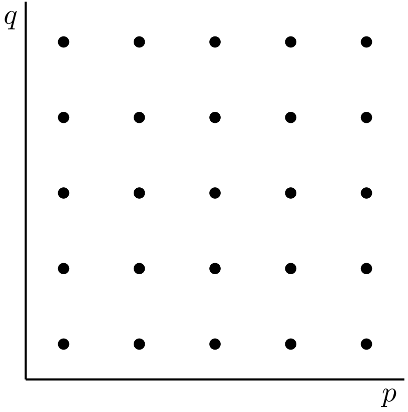
The differentials of E_{r+1} are not determined by (E_r, d_r).





Convergence of Spectral Sequences

In many spectral sequences the pages *converge* to $E_\infty^{p,q}$.



$E_\infty^{p,q}$

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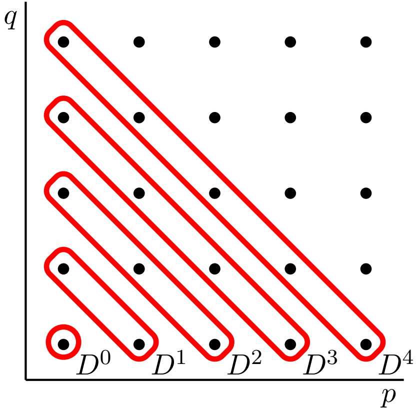
We can often compute the *abutment*, a "twisted sum" of the diagonals.



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For a bigraded abelian group $C^{p,q}$ and graded abelian group D^n we write

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- The spectral sequence converges to $E_\infty^{p,q}$;
- The abutment D^n is a twisted sum of the $E_\infty^{p,q}$ for $n = p + q$. This means that there are groups $(D^{n,q})_q$ and short exact sequences:

$$\begin{gathered}
0 \to E_\infty^{n,0} \to D^n \to D^{n,1} \to 0 \\
\vdots \\
0 \to E_\infty^{p,q} \to D^{n,q} \to D^{n,q+1} \to 0 \\
0 \to E_\infty^{p-1,q+1} \to D^{n,q+1} \to D^{n,q+2} \to 0 \\
\vdots \\
0 \to E_\infty^{0,n} \to D^{n,n} \to 0
\end{gathered}$$

Serre Spectral Sequence

Theorem (Serre Spectral Sequence)

If $f : X \to B$ is any map and Y is a truncated spectrum, then we have a spectral sequence E with

$$E_2^{p,q} = H^p(B; \lambda b. H^q(\mathsf{fib}_f(b); Y)) \Rightarrow H^{p+q}(X; Y).$$

If $Y = HA$ and B is pointed simply connected, then we get:

$$E_2^{p,q} = H^p(B; H^q(F; A)) \Rightarrow H^{p+q}(X; A).$$

where F is the fiber of f at b_0.

Atiyah-Hirzebruch Spectral Sequence

Theorem (Atiyah-Hirzebruch Spectral Sequence)

If X is any type and $Y : X \to \Omega$-Spectrum is a family of k-truncated spectra over X, then we have a spectral sequence E with

$$E_2^{p,q} = H^p(X; \lambda x.\pi_{-q}(Yx)) \Rightarrow H^{p+q}(X; \lambda x.Yx).$$

Atiyah-Hirzebruch Spectral Sequence

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The Atiyah-Hirzebruch spectral sequence is also true if we replace all cohomologies by reduced cohomologies:

$$E_2^{p,q} = \widetilde{H}^p(X; \lambda x.\pi_{-q}(Yx)) \Rightarrow \widetilde{H}^{p+q}(X; \lambda x.Yx).$$

Formalization

- There is a full formalization of the Serre and Atiyah-Hirzebruch spectral sequences for cohomology in Lean.
- As an application, we formalized the Gysin sequence.
- Other applications are in progress.
- Available at `github.com/cmu-phil/Spectral`.
- Formalized by vD, Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke and Mike Shulman.

Homology Spectral Sequences

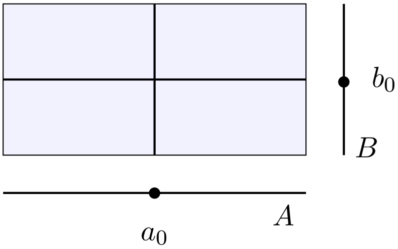
For many applications, we also need the Serre spectral sequence for homology.

For example, the version for homology gives Hurewicz theorem.

Smash Product

For pointed types A and B, the smash product $A \wedge B$ is the following homotopy pushout.

$$\begin{array}{ccc} A \vee B & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & A \wedge B \end{array}$$



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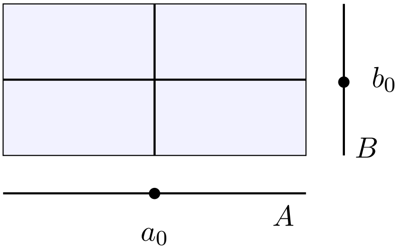

Given a type X and prespectrum Y we can now define a prespectrum $X \wedge Y$ with

$$(X \wedge Y)_n :\equiv X \wedge Y_n.$$

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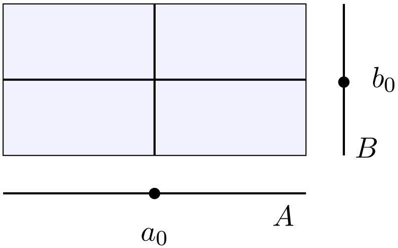

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The homology of X with coefficients in a prespectrum Y can be defined as

$$Y_n(X) \equiv \widetilde{H}_n(X; Y) :\equiv \pi_n(X \wedge Y) = \mathsf{colim}_k(\pi_{n+k}(X \wedge Y_k)).$$

Symmetric Monoidal Structure

Last HoTTEST Guillaume talked about his approach to prove that $\wedge$ form a 1-coherent symmetric monoidal product on pointed types.

Together with Stefano Piceghello I have tried to prove this using the adjunction $(-) \wedge B \dashv B \to^* (-)$, i.e.

$$(A \wedge B \to^* C) \simeq^* (A \to^* B \to^* C).$$

We have formalized this adjunction, natural in A, B and C.

This gives us associativity, symmetry and $\Sigma(A \wedge B) \simeq^* A \wedge \Sigma B$ as pointed natural equivalences.

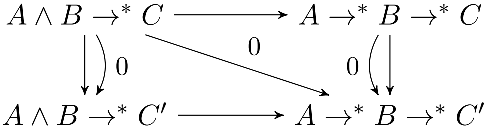
However, for the coherences (like the pentagon and hexagon) we need an enriched adjunction [Eilenberg-Kelly, Closed Categories, 1965].

Symmetric Monoidal Structure

The naturality of the adjunction is the following statement: Given $f : A' \to A$ and $g : B' \to B$ and $h : C \to C'$, the following square of pointed maps commutes:

$$\begin{array}{ccc} A \wedge B \to^* C & \longrightarrow & A \to^* B \to^* C \\ \downarrow & & \downarrow \\ A' \wedge B' \to^* C' & \longrightarrow & A' \to^* B' \to^* C' \end{array}$$

An enriched adjunction is one where the *proof of naturality* is pointed in h. That is, if $h \equiv 0_{C,C'}$ then the proof of naturality would be equal to the filler of the following square



Spectrification

If $X : \mathrm{Type}^*$ and $Y : \Omega$-Spectrum then $X \wedge Y$ is not generally an Ω-spectrum.

However, we can use the spectrification $L : \mathrm{Prespectrum} \to \Omega$-Spectrum.

L is a left adjoint to the forgetful map $U : \Omega$-Spectrum $\to \mathrm{Prespectrum}$.

It can be either defined as a family of recursive HITs, or as a colimit

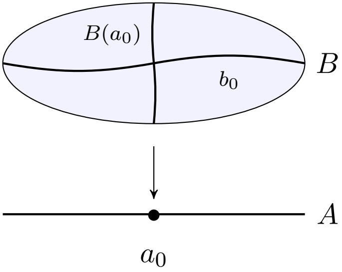
$$(LY)_n :\equiv \mathsf{colim}_k(\Omega^k Y_{n+k}).$$

With neither definition the adjunction has been carefully shown.

Parametrized Homology

We will also need parametrized homology.
$(x : A) \wedge B(x)$ is a parametrized version of the smash product, the following homotopy pushout:

$$\begin{array}{ccc} A \vee B(a_0) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \Sigma(x : A),\, B(x) & \longrightarrow & (x : A) \wedge B(x) \end{array}$$



Parametrized Homology

This has the following universal property:

$$((x : A) \wedge B(x) \to^* C) \simeq^* (\Pi^*(x : A),\ B(x) \to^* C).$$

Therefore,

$$\Sigma((x : A) \wedge B(x)) \simeq^* (x : A) \wedge \Sigma B(x).$$

This means that we can define $(x : A) \wedge Y x$ for $A : \mathrm{Type}^*$ and $Y : A \to \Omega$-Spectrum.

We define parametrized homology as

$$\widetilde{H}_n(X; \lambda x.Y x) :\equiv \pi_n((x : X) \wedge Y x).$$

Sequence of Spectra

We use the following result to prove the Atiyah-Hirzebruch theorem. This is a stable analogue of the Bousfield-Kan spectral sequence.

Theorem

Given a sequence of spectra

$$\cdots \to A_s \xrightarrow{f_s} A_{s-1} \xrightarrow{f_{s-1}} A_{s-2} \to \cdots$$

with fibers $F_s :\equiv \mathrm{fib}_{f_s}$, *suppose for all* n

- $\pi_n(A_s) = 0$ *for* s *small enough*
- $\pi_n(f_s)$ *is an isomorphism for* s *large enough.*

Then we have a spectral sequence E *with*

$$E_2^{n,s} = \pi_n(F_s) \Rightarrow \pi_n(A_\infty).$$

For cohomology we apply this using $A_s :\equiv \Pi^*(x : X), \|Yx\|_s$.

For homology, can we replace dependent maps by parametrized smash?

Spectral Sequences for Homology

Given $X : \mathrm{Type}^*$ and $Y : X \to \mathrm{Prespectrum}$. We can form:

$$\cdots \to \|Yx\|_s \to \|Yx\|_{s-1} \to \cdots$$

Spectral Sequences for Homology

Given $X : \mathrm{Type}^*$ and $Y : X \to \mathrm{Prespectrum}$. We can form:

$$\cdots \to (x : X) \wedge \|Yx\|_s \to (x : X) \wedge \|Yx\|_{s-1} \to \cdots$$

Spectral Sequences for Homology

Given $X : \mathrm{Type}^*$ and $Y : X \to \mathrm{Prespectrum}$. We can form:

$$\cdots \to L((x : X) \wedge \|Yx\|_s) \to L((x : X) \wedge \|Yx\|_{s-1}) \to \cdots$$

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To compute the fiber of this map we need to prove that smashing preserves fiber sequences.

- For spectra, the map $X \vee Y \to X \times Y$ is ∞-connected.
- Whitehead's theorem implies that fiber sequences and cofiber sequences are the same.
- We only want the correct homotopy groups, so we can probably avoid the use of Whitehead's Theorem.

Spectral Sequences for Homology

To actually get the Serre spectral sequence we might need a weaker notion of convergence than the one used for cohomology.

If we overcome these challenges, we get for a family Y of prespectra: (AHSS)

$$E^2_{p,q} = \widetilde{H}_p(X; \lambda x.\pi_q(Yx)) \Rightarrow \widetilde{H}_{p+q}(X; \lambda x.Yx).$$

For $f : X \to B$ and a prespectrum Y: (SSS)

$$E^2_{p,q} = H_p(B; \lambda b.H_q(\mathsf{fib}_f(b); Y)) \Rightarrow H_{p+q}(X; Y).$$

Applications

Corollary (Gysin sequence)

If $f : E \to^* B$ is a pointed map with fiber $\mathsf{fib}_f(b_0) \simeq^* \mathbb{S}^{n-1}$ for $n \geq 2$ and if B is simply connected and A is an abelian group, then there exists a long exact sequence

$$\cdots \to H^{i-1}(E; A) \to H^{i-n}(B; A) \to H^i(B; A) \to H^i(E; A) \to \cdots .$$

Corollary (Wang sequence)

If $f : E \to^* \mathbb{S}^n$ is a pointed map with fiber F for $n \geq 2$, then there exists a long exact sequence

$$\cdots \to H^{i-1}(F; A) \to H^{i-n}(F; A) \to H^i(E; A) \to H^i(F; A) \to \cdots .$$

They both also have analogues for homology.

Example: Gysin sequence

Given $\mathbb{S}^{n-1} \hookrightarrow E \xrightarrow{f} B$. Page 2 of the spectral sequence is

$$E_2^{p,q} = H^p(B; H^q(\mathbb{S}^{n-1}; A)) = \begin{cases} H^p(B; A) & \text{if } q \in \{0, n-1\} \\ 0 & \text{otherwise.} \end{cases}$$

$$\deg(d_r) = (r, -(r-1))$$

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q					
$n-1$	$H^0(B)$	$H^1(B)$	$\dots$	$H^n(B)$	$H^{n+1}(B)$
0	$H^0(B)$	$H^1(B)$	$\dots$	$H^n(B)$	$H^{n+1}(B)$
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$E_2^{p,q}$

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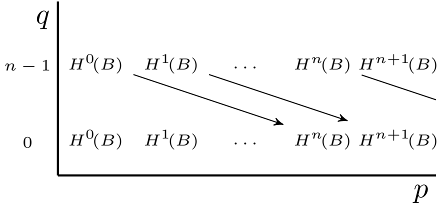

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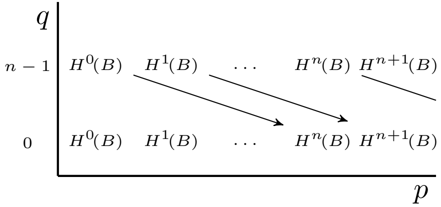

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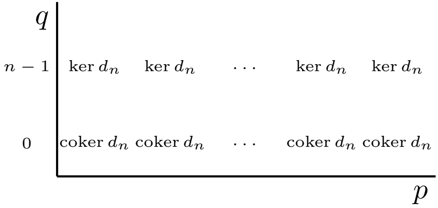

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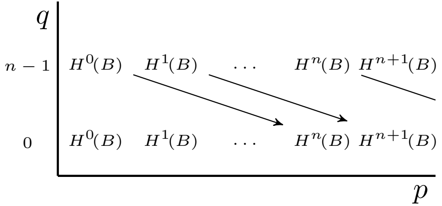

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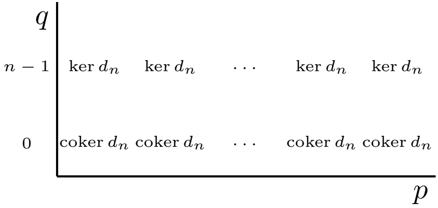
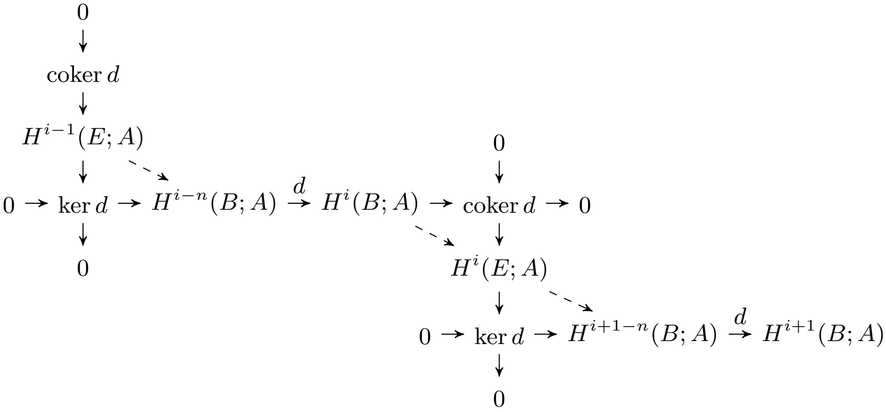


$E_{n+1}^{p,q} = E_\infty^{p,q}$

The abutment gives short exact sequences

$$0 \to \operatorname{coker} d_n^{i-n,n-1} \to H^i(E; A) \to \ker d_n^{i-(n-1),n-1} \to 0$$

Example: Gysin sequence



Future Applications

- Hurewicz theorem
- Serre class theorem: If $\mathcal{C}$ is a Serre class and X path connected and abelian then $\pi_n(X) \in \mathcal{C}$ for all n iff $H_n(X) \in \mathcal{C}$ for all n.
  Challenges:
  - The proof uses the Universal Coefficient Theorem, which might require the axiom of choice.
  - Constructively, the collection of finite abelian groups and the collection of finitely generated abelian groups do not form Serre classes.
- We can compute (co)homology groups of generalized cohomology theories (like K-theory).
- Computation of more homotopy groups of spheres.

Thank you