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MULTIQUADRATIC EXTENSIONS, RIGID FIELDS AND PYTHAGOREAN FIELDS

DAVID B. LEEP AND TARA L. SMITH

Abstract

Let F be a field of characteristic other than 2. Let $F^{(2)}$ denote the compositum over F of all quadratic extensions of F, let $F^{(3)}$ denote the compositum over $F^{(2)}$ of all quadratic extensions of $F^{(2)}$ that are Galois over F, and let $F^{\{3\}}$ denote the compositum over $F^{(2)}$ of all quadratic extensions of $F^{(2)}$. This paper shows that $F^{(3)} = F^{\{3\}}$ if and only if F is a rigid field, and that $F^{(3)} = K^{(3)}$ for some extension K of F if and only if F is Pythagorean and $K = F(\sqrt{-1})$. The proofs depend mainly on the behavior of quadratic forms over quadratic extensions, and the corresponding norm maps.

1. *Introduction*

Let F be a field of characteristic not 2. We consider the Galois extension $F^{(3)}$ of F obtained by first taking $F^{(2)}$ to be the compositum over F of all quadratic extensions of F, and then taking $F^{(3)}$ to be the compositum over $F^{(2)}$ of all the quadratic extensions of $F^{(2)}$ that are Galois over F. We also denote by $F^{\{3\}}$ the compositum over $F^{(2)}$ of all quadratic extensions of $F^{(2)}$. Thus $F^{\{3\}} = (F^{(2)})^{(2)}$.

We shall characterize those fields F with the property that $F^{(3)} = F^{\{3\}}$, and we shall also determine precisely when one can have a field extension K/F for which $F^{(3)} = K^{(3)}$. In fact, we shall prove the following two theorems.

Theorem A. *$F^{(3)} = F^{\{3\}}$ if and only if F is a rigid field.*

Theorem B. *Let K/F be a proper extension of fields. The following statements are equivalent:*
(1) *$F^{(3)} = K^{(3)}$;*
(2) *$F^{(2)} = K^{(2)}$;*
(3) *F is Pythagorean and $K = F(\sqrt{-1})$.*

Our proofs use only elementary methods from Galois theory. Although Theorem A is proved in [**1**, Theorem 3.1 and Definitions 2.2 and 2.3], and Theorem B is proved in [**6**, Theorems 3.2 and 3.6 and Corollary 3.7], these earlier proofs rely on a number of specialized and highly technical results, and are substantially longer than the proofs in this paper.

To begin with, we recall some notation and develop some useful results concerning the behavior of quadratic forms under quadratic and multiquadratic extensions of fields. In the second section, we develop a number of results on rigid elements and quadratic extensions, and provide a simple new proof that quadratic extensions of rigid fields remain rigid. In Sections 3 and 4 we prove Theorems A and B cited above.

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Throughout this paper, fields are assumed to have characteristic other than 2; $\dot{F}$ denotes the multiplicative group of nonzero elements of the field F, $D_F(q)$ denotes the set of nonzero elements represented over F by the quadratic form q, $N_{K/F}(\alpha)$ denotes the norm in F of an element $\alpha \in K$, and $\langle a_1, \ldots, a_n \rangle$ denotes the n-dimensional quadratic form $a_1x_1^2 + \ldots + a_nx_n^2$.

Definition 1.1. An element $a \in \dot{F} \backslash \pm \dot{F}^2$ is said to be *rigid* if $D_F(\langle 1, a \rangle) = \dot{F}^2 \cup a\dot{F}^2$. The field F is said to be a *rigid field* (or a *C-field*) if every $a \in \dot{F} \backslash \pm \dot{F}^2$ is rigid.

Definition 1.2. A field F is *Pythagorean* if every sum of squares is a square.

Proposition 1.3 (Square class exact sequence [**4**, Theorem 3.4, p. 202]). *Let $K = F(\sqrt{a})$ be a quadratic extension of the field F. Let $\epsilon : \dot{F}/\dot{F}^2 \longrightarrow \dot{K}/\dot{K}^2$ be the map induced by the inclusion of F in K, and let $N : \dot{K}/\dot{K}^2 \longrightarrow \dot{F}/\dot{F}^2$ be the homomorphism induced by the norm from K to F. Then the following sequence is exact:*

$$1 \longrightarrow \{\dot{F}^2, a\dot{F}^2\} \longrightarrow \dot{F}/\dot{F}^2 \xrightarrow{\epsilon} \dot{K}/\dot{K}^2 \xrightarrow{N} \dot{F}/\dot{F}^2.$$

Corollary 1.4. *Keeping the notation of Proposition 1.3, the map $\epsilon : \dot{F}/\dot{F}^2 \longrightarrow \dot{K}/\dot{K}^2$ is surjective if and only if F is Pythagorean and $K = F(\sqrt{-1})$.*

Proof. The map ϵ is surjective if and only if $\ker(N) = \dot{K}/\dot{K}^2$, if and only if $D_F(\langle 1, -a \rangle) = \dot{F}^2$, and if and only if $a \in -\dot{F}^2$ and every sum of squares in F is a square. $\square$

The following proposition has appeared in several places; see [**7**, Lemma 1.14]. The proof by Berman [**2**, Lemma 3.5] is particularly nice. Here is a different proof.

Proposition 1.5. *Let $K = F(\sqrt{b})$. Then*

$$D_K(\langle 1, -a \rangle) \cap F = D_F(\langle 1, -a \rangle) D_F(\langle 1, -ab \rangle).$$

Proof. Since the result is trivial if b is a square in F, we may assume that $[K : F] = 2$. Let $\alpha \in D_K(\langle 1, -a \rangle)$; so

$$\alpha = (x + y\sqrt{b})^2 - a(z + w\sqrt{b})^2 = x^2 + by^2 - az^2 - abw^2 + 2(xy - azw)\sqrt{b}$$

for some $x, y, z, w \in F$. If $\alpha \in F$, then $xy - azw = 0$. First, assume that $x \neq 0$, so $y = azw/x$. Then $\alpha = x^2 + by^2 - az^2 - abw^2 = (x^2 - az^2)(1 - ab(w/x)^2)$. Next, assume that $x = 0$. Then either $z = 0$ or $w = 0$. If $z = 0$, then $by^2 - abw^2 = (w^2 - a(y/a)^2)(-ab)$. If $w = 0$, then $by^2 - az^2 = -a(z^2 - ab(y/a)^2)$. In all cases, we have $\alpha \in D_F(\langle 1, -a \rangle) D_F(\langle 1, -ab \rangle)$. The reverse inclusion is trivial. $\square$

Proposition 1.6 ([**3**, Theorem 2.1]). *Let $L = F(\sqrt{a_1}, \ldots, \sqrt{a_n})$, for $a_i \in \dot{F}$, $[L : F] = 2^n$, where $n \geqslant 1$. Let $\alpha \in \dot{L}$. Then $N_{L/K_i}(\alpha) \in \dot{K}_i^2$ for all intermediate fields $K_i \supseteq F$ with $[L : K_i] = 2$ if and only if $\alpha \in \dot{F}\dot{L}^2$.*

Proof. If $\alpha \in \dot{F}\dot{L}^2$, then it is clear that $N_{L/K_i}(\alpha) \in \dot{K}_i^2$. Now assume that $N_{L/K_i}(\alpha) \in \dot{K}_i^2$ for all intermediate fields $K_i \supseteq F$ with $[L : K_i] = 2$. We prove the result by induction on n. The case $n = 1$ is a consequence of the square class exact sequence. Now assume that $n \geqslant 2$.

Let $K = F(\sqrt{a_1}, \dots, \sqrt{a_{n-1}})$; so $L = K(\sqrt{a_n})$ and $[L : K] = 2$. We have $N_{L/K}(\alpha) \in \dot{K}^2$, and so $\alpha \in \dot{K}\dot{L}^2$ by the $n = 1$ case. Without loss of generality, we may take α to be in $\dot{K}$. Let $F \subseteq E \subseteq K$ with $[K : E] = 2$. Then $K = E(\sqrt{b})$, for $b \in \dot{F}$, and $L = E(\sqrt{a_n}, \sqrt{b})$. We have $[L : E(\sqrt{a_n})] = [L : E(\sqrt{a_n b})] = 2$ and

$$N_{K/E}(\alpha) = N_{L/E(\sqrt{a_n})}(\alpha) = N_{L/E(\sqrt{a_n b})}(\alpha) \in \dot{E} \cap E(\sqrt{a_n})^2 \cap E(\sqrt{a_n b})^2 = \dot{E}^2.$$

This holds for all $[K : E] = 2$, where $F \subseteq E$, and since $[K : F] = 2^{n-1}$, induction implies that $\alpha \in \dot{F}\dot{K}^2 \subseteq \dot{F}\dot{L}^2$. □

For the next two lemmas, the following notation will be in effect. We let L/F be a finite multiquadratic extension, where $[L : F] = 2^n \geqslant 4$, and assume that $F \subset F(\sqrt{-1}) \subset L$, where '$\subset$' denotes strict inclusion. We set $G = \mathrm{Gal}(L/F)$ and $H = \mathrm{Gal}(L/F(\sqrt{-1}))$, and write $G = H \cup \tau H$. Let $\alpha \in \dot{L}$ and assume that $L(\sqrt{\alpha})/F(\sqrt{-1})$ is Galois, where $[L(\sqrt{\alpha}) : L] = 2$. Then we have $\sigma(\alpha) \cdot \alpha \in \dot{L}^2$ for all $\sigma \in H$.

Lemma 1.7. *$L(\sqrt{\alpha}, \sqrt{\tau(\alpha)})$ is the Galois closure of $L(\sqrt{\alpha})/F$, and $L(\sqrt{\tau(\alpha)})/F(\sqrt{-1})$ is Galois.*

Proof. The Galois closure of $L(\sqrt{\alpha})/F$ is $L(\{\sqrt{\sigma(\alpha)} \mid \sigma \in G\})$, which equals $L(\sqrt{\alpha}, \sqrt{\tau(\alpha)})$, since $\sigma(\alpha) \cdot \alpha \in \dot{L}^2$ for all $\sigma \in H$, and $\tau(\alpha) \cdot \tau\sigma(\alpha) = \tau(\sigma(\alpha) \cdot \alpha) \in \tau(\dot{L}^2) = \dot{L}^2$ for all $\sigma \in H$. Thus $\sigma(\alpha) \equiv \alpha$ or $\sigma(\alpha) \equiv \tau(\alpha) \bmod \dot{L}^2$ for all $\sigma \in G$. For the second claim, observe that $\sigma(\tau(\alpha)) \cdot \tau(\alpha) = \tau(\sigma(\alpha) \cdot \alpha) \in \tau(\dot{L}^2) = \dot{L}^2$ for all $\sigma \in H$. □

Lemma 1.8. *$\tau(\alpha) \cdot \alpha \in F(\dot{\sqrt{-1}})\dot{L}^2$.*

Proof. Let $[L : K] = 2$ where $F(\sqrt{-1}) \subseteq K \subseteq L$, and let $\mathrm{Gal}(L/K) = \{1, \sigma_K\} \subseteq H$. Write $L = K(\sqrt{b})$, for $b \in \dot{F}$. Then $\sigma_K(\alpha) \cdot \alpha \in \dot{K} \cap L^2 = \dot{K}^2 \cup b\dot{K}^2$. Since K/F is Galois and $b \in \dot{F}$, we have $\tau(\dot{K}^2) = \dot{K}^2$ and $\tau(b\dot{K}^2) = b\dot{K}^2$. Therefore $\sigma_K(\tau(\alpha) \cdot \alpha)(\tau(\alpha) \cdot \alpha) = \tau(\sigma_K(\alpha) \cdot \alpha)(\sigma_K(\alpha) \cdot \alpha) \in \dot{K}^2$ for every K with $[L : K] = 2$ and $F(\sqrt{-1}) \subseteq K$. Then by Proposition 1.6 it follows that $\tau(\alpha) \cdot \alpha \in F(\dot{\sqrt{-1}})\dot{L}^2$. □

Lemma 1.9. *Let M/k be a finite Galois extension, and suppose that there exists a chain of fields $k \subseteq E \subseteq F \subseteq K \subseteq L \subseteq M$ such that the following statements hold:*

(1) *$[L : K] = [K : F] = [F : E] = 2$;*
(2) *L/F is Galois with $\mathrm{Gal}(L/F) \cong \mathbb{Z}/4\mathbb{Z}$;*
(3) *K/E is Galois with $\mathrm{Gal}(K/E) \cong \mathbb{Z}/4\mathbb{Z}$.*

Then $\mathrm{Gal}(M/k)$ contains an element of order 8.

Proof. There exists $\sigma \in \mathrm{Gal}(M/E)$ such that $\sigma|_K$ has order 4, since $\mathrm{Gal}(K/E) \cong \mathbb{Z}/4\mathbb{Z}$. Because $[\mathrm{Gal}(M/E) : \mathrm{Gal}(M/F)] = [F : E] = 2$, it follows that $\sigma^2 \in \mathrm{Gal}(M/F)$. Then $\sigma^2|_L$ is an element of $\mathrm{Gal}(L/F)$, which is not the identity on K (since $\sigma|_K$ has order 4). Therefore $\sigma^2|_L$ is an element of order 4. This implies that $\sigma|_L$ has order 8. Therefore the order of σ is divisible by 8, and $\mathrm{Gal}(M/E)$ (and hence also $\mathrm{Gal}(M/k)$) contains an element of order 8. □

2. *Rigid fields*

We prove several general results concerning the preservation of rigidity under quadratic extensions.

PROPOSITION 2.1. *Let* $L = F(\sqrt{a},\sqrt{b})$, $[L : F] = 4$. *If* $-a$ *and* $-b$ *are rigid in* F, *then* $-a$ *and* $-b$ *remain rigid in* $F(\sqrt{ab})$.

Proof. We know that $-a \notin \pm\dot{F}^2$ and $-b \notin \pm\dot{F}^2$ by the definition of 'rigid'. Suppose that $-a \in \pm\dot{F(\sqrt{ab})}^2$. Then $\pm a \in F(\sqrt{ab})^2 \cap F = F^2 \cup abF^2$. We know that $\pm a \notin F^2$, and if $\pm a \in abF^2$, then $\pm b \in F^2$, a contradiction. Thus $-a \notin \pm F(\sqrt{ab})^2$.

Let $A \in D_{F(\sqrt{ab})}(\langle 1,-a\rangle)$. Then $A = N_{L/F(\sqrt{ab})}(\alpha)$ for some $\alpha \in \dot{L}$. Then

$$N_{L/F}(\alpha) = N_{F(\sqrt{a})/F}(N_{L/F(\sqrt{a})}(\alpha)) \in D_F(\langle 1,-a\rangle) = \dot{F}^2 \cup -a\dot{F}^2,$$
$$N_{L/F}(\alpha) = N_{F(\sqrt{b})/F}(N_{L/F(\sqrt{b})}(\alpha)) \in D_F(\langle 1,-b\rangle) = \dot{F}^2 \cup -b\dot{F}^2.$$

Thus $N_{L/F}(\alpha) \in (\dot{F}^2 \cup -a\dot{F}^2) \cap (\dot{F}^2 \cup -b\dot{F}^2) = \dot{F}^2$. It follows that

$$N_{F(\sqrt{ab})/F}(A) = N_{F(\sqrt{ab})/F}(N_{L/F(\sqrt{ab})}(\alpha)) = N_{L/F}(\alpha) \in \dot{F}^2.$$

This implies that $A = cB^2$, where $c \in \dot{F}$ and $B \in F(\sqrt{ab})$. Then Proposition 1.5 implies that

$$\begin{aligned} c \in D_{F(\sqrt{ab})}(\langle 1,-a\rangle) \cap \dot{F} &= D_F(\langle 1,-a\rangle) \cdot D_F(\langle 1,-b\rangle) \\ &= (\dot{F}^2 \cup -a\dot{F}^2)(\dot{F}^2 \cup -b\dot{F}^2) \\ &= \dot{F}^2 \cup -a\dot{F}^2 \cup -b\dot{F}^2 \cup ab\dot{F}^2 \\ &\subseteq \dot{F(\sqrt{ab})}^2 \cup -a\dot{F(\sqrt{ab})}^2. \end{aligned}$$

This shows that $D_{F(\sqrt{ab})}(\langle 1,-a\rangle) = \dot{F(\sqrt{ab})}^2 \cup -a\dot{F(\sqrt{ab})}^2$, and hence $-a$ is rigid in $F(\sqrt{ab})$. By symmetry (or by observing that $(-a)(-b) \in F(\sqrt{ab})^2$), we see that $-b$ is also rigid in $F(\sqrt{ab})$. □

COROLLARY 2.2. *Let* F *be a rigid field, and let* $K = F(\sqrt{d})$, *where* $[K : F] = 2$. *Let* $a \in \dot{F}$, *and assume that* $a \notin \pm\dot{K}^2$. *Then* a *is rigid in* K.

Proof. Since $[K : F] = 2$, we know that $d \notin \dot{F}^2$, and since $a \notin \pm\dot{K}^2$, we have $a \notin \dot{F}^2 \cup -\dot{F}^2 \cup d\dot{F}^2 \cup -d\dot{F}^2$. We now check the hypotheses of Proposition 2.1, substituting $-a$ for a and $-ad$ for b. We see that $[F(\sqrt{-a},\sqrt{-ad}) : F] = 4$, because $-a \notin \dot{F}^2$ and $-ad \notin \dot{F}^2 \cup -a\dot{F}^2$. Since $a \notin \pm\dot{F}^2$ and $ad \notin \pm\dot{F}^2$, the rigidity of F implies that a and ad are rigid in F. Now, by Proposition 2.1, a is rigid in $F(\sqrt{d})$. □

PROPOSITION 2.3. *Suppose that* a *is rigid in* F, *and that* $-1 \notin \dot{F}^2$. *If we have* $D_F(\langle 1,1\rangle) \subseteq F^2 \cup -F^2$, *then* $D_{F(\sqrt{a})}(\langle 1,1\rangle) \subseteq \dot{F(\sqrt{a})}^2 \cup -\dot{F(\sqrt{a})}^2$.

Proof. The proof is analogous to that for Proposition 2.1, replacing a with -1 and b with $-a$. Since $a \notin \pm\dot{F}^2$ and $-1 \notin \dot{F}^2$, it follows that $[F(\sqrt{-1},\sqrt{-a}) : F] = 4$.

Let $L = F(\sqrt{-1},\sqrt{-a})$, and let $A \in D_{F(\sqrt{a})}(\langle 1,1\rangle)$. Then $A = N_{L/F(\sqrt{a})}(\alpha)$ for some $\alpha \in \dot{L}$.

As before,

$$N_{L/F}(\alpha) = N_{F(\sqrt{-1})/F}(N_{L/F(\sqrt{-1})}(\alpha)) \in D_F(\langle 1,1\rangle) \subseteq F^2 \cup -F^2,$$
$$N_{L/F}(\alpha) = N_{F(\sqrt{-a})/F}(N_{L/F(\sqrt{-a})}(\alpha)) \in D_F(\langle 1,a\rangle) = \dot{F}^2 \cup a\dot{F}^2.$$

Thus

$$N_{L/F}(\alpha) \in (F^2 \cup -F^2) \cap (\dot{F}^2 \cup a\dot{F}^2) = \dot{F}^2.$$

It follows that $N_{F(\sqrt{a})/F}(A) = N_{L/F}(\alpha) \in \dot{F}^2$, and this implies that $A = cB^2$, where $c \in \dot{F}$ and $B \in F(\sqrt{a})$. Then Proposition 1.5 implies that

$$\begin{aligned} c \in D_{F(\sqrt{a})}(\langle 1,1\rangle) \cap F &= D_F(\langle 1,1\rangle) \cdot D_F(\langle 1,a\rangle) \\ &\subseteq (F^2 \cup -F^2)(F^2 \cup aF^2) \\ &= F^2 \cup aF^2 \cup -F^2 \cup -aF^2 \\ &\subseteq F(\sqrt{a})^2 \cup -F(\sqrt{a})^2. \end{aligned}$$

Therefore, $D_{F(\sqrt{a})}(\langle 1,1\rangle) \subseteq F(\sqrt{a})^2 \cup -F(\sqrt{a})^2$. $\square$

Remark 2.4. If F is a rigid field with $-1 \notin \dot{F}^2$, then $D_F(\langle 1,1\rangle) \subseteq \dot{F}^2 \cup -\dot{F}^2$. For if $b \in D_F(\langle 1,1\rangle)$, then $\langle 1,1\rangle \simeq \langle b,b\rangle$, and so $\langle 1,-b\rangle \simeq \langle -1,b\rangle$. If $b \notin \pm\dot{F}^2$, then $-b$ is rigid and $D_F(\langle 1,-b\rangle) = \dot{F}^2 \cup -b\dot{F}^2$. This then implies that $b \in \dot{F}^2 \cup -b\dot{F}^2$, a contradiction.

The following result has been proved by Ware [**8**], using different methods. The proof that we give here uses only the square class exact sequence and the elementary results on preservation of rigidity under quadratic extensions proved above. A 2-extension of F is a field obtained from F by a sequence of quadratic extensions.

Theorem 2.5. *Let F be a rigid field. Then every 2-extension of F is rigid.*

Proof. It suffices to show that every quadratic extension of a rigid field is rigid. Let $K = F(\sqrt{d})$ be a proper quadratic extension of F. If $d \notin \pm\dot{F}^2$, then $-d$ is rigid in F and $D_F(\langle 1,-d\rangle) = \dot{F}^2 \cup -d\dot{F}^2$. If $d \in -\dot{F}^2$, then $-1 \notin \dot{F}^2$, and it follows that $D_F(\langle 1,1\rangle) \subseteq \dot{F}^2 \cup -\dot{F}^2$ by Remark 2.4. Thus, in either case, the square class exact sequence gives $|\operatorname{im} N| \leqslant 2$. This implies that

$$|\dot{K}/\dot{F}\dot{K}^2| = |(\dot{K}/\dot{K}^2)/\operatorname{im}\epsilon| = |(\dot{K}/\dot{K}^2)/\ker N| = |\operatorname{im} N| \leqslant 2.$$

Let $\alpha \in \dot{K}$, but $\alpha \notin \pm\dot{K}^2$. We shall show that α is rigid in K.

If $\alpha \in \dot{F}\dot{K}^2$, then $\alpha = a\gamma^2$, where $a \in \dot{F}$ and $\gamma \in \dot{K}$. Since $a \notin \pm\dot{K}^2$, Corollary 2.2 implies that a (and hence α) is rigid in K. Thus if $\dot{K} = \dot{F}\dot{K}^2$, then K is rigid.

Now assume that $\alpha \notin \dot{F}\dot{K}^2$. Then $\dot{K} = \dot{F}\dot{K}^2 \cup \alpha\dot{F}\dot{K}^2$, with $\alpha \notin \pm\dot{K}^2$. We must show that $D_K(\langle 1,\alpha\rangle) = \dot{K}^2 \cup \alpha\dot{K}^2$. Let $\beta \in D_K(\langle 1,\alpha\rangle)$.

First, suppose that $\beta \in \dot{F}\dot{K}^2$. We may assume $\beta = b \in \dot{F}$. If $-b \notin \pm\dot{K}^2$, then $-b$ is rigid in K by Corollary 2.2. Hence $-\alpha \in D_K(\langle 1,-b\rangle) = \dot{K}^2 \cup -b\dot{K}^2 \subseteq \dot{F}\dot{K}^2$, contradicting our assumption. Therefore $-b \in \pm\dot{K}^2$. If $b \in \dot{K}^2$, we have finished, so we may assume that $-b \in \dot{K}^2$ with $-1 \notin \dot{K}^2$. Then $d \notin \pm\dot{F}^2$, so d is rigid in F. We have $-\alpha \in D_K(\langle 1,-b\rangle) = D_K(\langle 1,1\rangle) \subseteq \dot{K}^2 \cup -\dot{K}^2$ by Proposition 2.3. But this means that $\alpha \in \dot{F}\dot{K}^2$, again giving a contradiction.

Now suppose that $\beta \notin \dot{F}\dot{K}^2$. Then $\beta \in \alpha\dot{F}\dot{K}^2$, and we can assume that $\beta = b\alpha$, for $b \in \dot{F}$. Since $b\alpha = \beta \in D_K(\langle 1,\alpha\rangle)$, it follows that $b \in D_K(\langle 1,\alpha\rangle)$. The argument above then implies that $b \in \dot{K}^2$, so $\beta = b\alpha \in \alpha\dot{K}^2$. Thus $D_K(\langle 1,\alpha\rangle) \subseteq \dot{K}^2 \cup \alpha\dot{K}^2$ for all $\alpha \in \dot{K}$, where $\alpha \notin \pm\dot{K}^2$. This implies each such α is rigid in K, and so K is a rigid field. $\square$

3. *Proof of Theorem A*

We prove the forward direction of Theorem A in Theorem 3.1, and the reverse direction in Theorem 3.3.

THEOREM 3.1. *If $F^{(3)} = F^{\{3\}}$, then F is a rigid field.*

Proof. Assume that $F^{(3)} = F^{\{3\}}$. Then for all $\alpha \in F^{(2)}$, $F^{(2)}(\sqrt{\alpha})$ is Galois over F. Thus for all $\sigma \in \mathrm{Gal}(F^{(2)}/F)$, $\sigma(\alpha) \cdot \alpha \in (F^{(2)})^2$. Choose $a, b \in \dot{F}\backslash\dot{F}^2$, independent mod $\dot{F}^2$, with $a \notin -\dot{F}^2$. We must show that $b \notin D_F(\langle 1, a\rangle)$. Choose $\sigma \in \mathrm{Gal}(F^{(2)}/F)$ such that $\sigma(\sqrt{a}) = \sqrt{a}$ and $\sigma(\sqrt{b}) = -\sqrt{b}$, and let K be the fixed field of σ. Then $F^{(2)} = K(\sqrt{b})$ and $\sqrt{a} \in K$.

Since $a \notin -\dot{F}^2$, we observe that $\sqrt{a} \notin (F^{(2)})^2$, for otherwise $F(\sqrt{-1}, \sqrt[4]{a}) \subseteq F^{(2)}$ and $F(\sqrt{-1}, \sqrt[4]{a})/F(\sqrt{-1})$ would be a cyclic quartic extension, which is impossible.

Suppose that $b \in D_F(\langle 1, a\rangle)$. Then there exist $x, y, z \in F$, not all zero, such that $x^2 + ay^2 - bz^2 = 0$. Let $\alpha = x + y\sqrt{a} + z\sqrt{b}$. Then

$$0 \neq \sigma(\alpha) \cdot \alpha = x^2 + ay^2 - bz^2 + 2xy\sqrt{a} = 2xy\sqrt{a} \in (F^{(2)})^2.$$

Since $F \subseteq (F^{(2)})^2$, it follows that $\sqrt{a} \in (F^{(2)})^2$, a contradiction. □

LEMMA 3.2. *If K is a finite multiquadratic extension of a rigid field F and $\sqrt{-1} \in K$, then every quadratic extension of K is Galois over F.*

Proof. Let $\alpha \in K$, and let $K(\sqrt{\alpha})$ be a quadratic extension of K. We must show that $\sigma(\alpha) \cdot \alpha \in K^2$ for all $\sigma \in \mathrm{Gal}(K/F)$. This is trivially true if $\sigma = 1$.

Let $1 \neq \sigma \in \mathrm{Gal}(K/F)$, and let E be the fixed field of σ. Then $K = E(\sqrt{a})$ for some $a \in \dot{F}$, and $\sigma(\alpha) \cdot \alpha \in D_E(\langle 1, -a\rangle)$. We shall show that $D_E(\langle 1, -a\rangle) \subseteq \dot{E}^2 \cup a\dot{E}^2 \subseteq \dot{K}^2$. Since E is a 2-extension of F, Theorem 2.5 implies that E is a rigid field. If $-1 \in \dot{E}^2$, then $a \notin \pm\dot{E}^2$, and $D_E(\langle 1, -a\rangle) = D_E(\langle 1, a\rangle) = \dot{E}^2 \cup a\dot{E}^2$ by the rigidity of E. If $-1 \notin \dot{E}^2$, we may take $a = -1$ since $-1 \in \dot{K}^2$. Since E is rigid, Remark 2.4 implies that $D_E(\langle 1, 1\rangle) \subseteq \dot{E}^2 \cup -\dot{E}^2$. □

THEOREM 3.3. *If F is a rigid field, then $F^{(3)} = F^{\{3\}}$.*

Proof. We must show that $F^{(2)}(\sqrt{\alpha})/F$ is a Galois extension for all $\alpha \in F^{(2)}$. Let $K = F(\sqrt{-1}, \alpha) \subseteq F^{(2)}$. Then K is a finite multiquadratic extension of F containing $\sqrt{-1}$, and hence Lemma 3.2 implies that $K(\sqrt{\alpha})/F$ is a Galois extension. Since $F^{(2)}/F$ is a Galois extension, it follows that $F^{(2)}(\sqrt{\alpha})/F$ is a Galois extension. □

COROLLARY 3.4. *Assume that $F^{(3)} = F^{\{3\}}$. Then $K^{(3)} = K^{\{3\}}$ for any 2-extension K of F.*

Proof. This follows immediately from Theorems 3.1, 2.5, and 3.3. □

4. *Proof of Theorem B*

Observe first that if F is Pythagorean and $K = F(\sqrt{-1})$, then Corollary 1.4 implies that $F^{(2)} = K^{(2)}$. Conversely, suppose that $F^{(2)} = K^{(2)}$. Then K is a proper multiquadratic extension of F, and the induced map $\epsilon : \dot{F}/\dot{F}^2 \longrightarrow \dot{K}/\dot{K}^2$ is surjective. Let $F \subseteq E \subseteq K$ where $[K : E] = 2$. Then the map $\dot{E}/\dot{E}^2 \longrightarrow \dot{K}/\dot{K}^2$ is surjective, and Corollary 1.4 implies that E is Pythagorean and $K = E(\sqrt{-1})$. But $F^{(2)} \subseteq E^{(2)} \subseteq K^{(2)}$, and so $F^{(2)} = E^{(2)}$. If $F \neq E$, the same reasoning would show that $-1 \in E^2$, a contradiction. Therefore $F = E$, and so F is Pythagorean and $K = F(\sqrt{-1})$. We have now proved the equivalence of conditions (2) and (3) of Theorem B. Theorems 4.2 and 4.3 below give the equivalence of conditions (1) and (3) of Theorem B.

LEMMA 4.1. $\mathrm{Gal}(F^{(3)}/F)$ *has exponent at most* 4.

Proof. Let $\sigma \in \mathrm{Gal}(F^{(3)}/F)$. Then $\sigma^2 \in \mathrm{Gal}(F^{(3)}/F^{(2)})$ since σ^2 fixes all square roots of elements in F. Since $\mathrm{Gal}(F^{(3)}/F^{(2)})$ has exponent at most 2, it follows that $\mathrm{Gal}(F^{(3)}/F)$ has exponent at most 4. $\square$

THEOREM 4.2. *Assume that F is Pythagorean, and that $K = F(\sqrt{-1})$. Then $F^{(3)} = K^{(3)}$.*

Proof. We may assume that $K \neq F$. Clearly, $F^{(3)} \subseteq K^{(3)}$, and the hypothesis implies that $F^{(2)} = K^{(2)}$, by Corollary 1.4. Suppose that $\alpha \in \dot{F}^{(2)}$ with $F^{(2)}(\sqrt{\alpha})/K$ Galois. We need to show that $F^{(2)}(\sqrt{\alpha})/F$ is Galois. In other words, we must see that $\sigma(\alpha)\cdot\alpha \in (F^{(2)})^2$ for all $\sigma \in \mathrm{Gal}(F^{(2)}/F)$. But $(F^{(2)})^2 = F(F^{(2)})^2 = F(\sqrt{-1})(F^{(2)})^2$ (by Corollary 1.4), so it is sufficient to show that $\sigma(\alpha)\cdot\alpha \in F(\sqrt{-1})(F^{(2)})^2$.

Let $L = F(\alpha,\sqrt{-1})$, so $L \subseteq F^{(2)}$ is a finite multiquadratic extension of F. If $[L : F] = 2$, then $\alpha \in F(\sqrt{-1})$, so $\sigma(\alpha)\cdot\alpha \in F(\sqrt{-1})$ for all $\sigma \in \mathrm{Gal}(F^{(2)}/F)$, and we have finished in this case. Otherwise $[L : F] \geqslant 4$, and we can apply the results of Lemmas 1.7 and 1.8 above to conclude that $\sigma(\alpha)\cdot\alpha \in F(\sqrt{-1})L^2 \subseteq F(\sqrt{-1})(F^{(2)})^2$ for all $\sigma \in \mathrm{Gal}(F^{(2)}/F)$. $\square$

THEOREM 4.3. *Let K/F be a proper extension of fields for which $F^{(3)} = K^{(3)}$. Then F is Pythagorean and $K = F(\sqrt{-1})$.*

Proof. We first reduce the statement to the case where K is a quadratic extension of F. Since $K \subseteq F^{(3)}$, there exists a quadratic extension $F \subseteq F(\sqrt{a}) \subseteq K$. Since $F^{(3)} \subseteq F(\sqrt{a})^{(3)} \subseteq K^{(3)} = F^{(3)}$, it follows that $F^{(3)} = F(\sqrt{a})^{(3)}$. Suppose that we can show that this implies that F is Pythagorean and $a \in -\dot{F}^2$. Since $F(\sqrt{a})^{(3)} = K^{(3)}$, the same reasoning would imply that K cannot be a proper extension of $F(\sqrt{a})$, and so $K = F(\sqrt{a})$. We now show that if either $a \notin -\dot{F}^2$ or F is not Pythagorean, we can produce an extension of K which is in $K^{(3)}$ but not in $F^{(3)}$.

First, suppose that $K = F(\sqrt{a})$, $[K : F] = 2$, and assume that $a \notin -\dot{F}^2$. Let $f(x) = x^8 - a \in F[x]$. If $\sqrt[8]{a}$ denotes an arbitrary root of $f(x)$, then $M = F(\sqrt[8]{a},\sqrt{-1},\sqrt{2})$ is a splitting field of f over F. Consider the following tower of extensions:

$$F \subseteq F' = F(\sqrt{-1}) \subseteq F'(\sqrt{a}) \subseteq F'(\sqrt[4]{a}) \subseteq F'(\sqrt[8]{a}) \subseteq M.$$

We have $[F'(\sqrt{a}) : F'] = 2$ since $\sqrt{a} \in F(\sqrt{-1})$ would imply that $a \in \dot{F}^2 \cup -\dot{F}^2$, contrary to the assumption. Since $x^2 - a$ is irreducible over F', it follows that both $x^4 - a$ and $x^8 - a$ are irreducible over F' [**5**, Theorem 9.1, p. 297]. Therefore $[F'(\sqrt[4]{a}) : F'] = 4$ and $[F'(\sqrt[8]{a}) : F'] = 8$. We have $\mathrm{Gal}(F'(\sqrt[4]{a})/F') \cong \mathbb{Z}/4\mathbb{Z}$, since $\sqrt{-1} \in F'$. Also, $\mathrm{Gal}(F'(\sqrt[8]{a})/F'(\sqrt{a})) \cong \mathbb{Z}/4\mathbb{Z}$, since $\sqrt[8]{a} = \sqrt[4]{\sqrt{a}}$ and $\sqrt{-1} \in F'(\sqrt{a})$. Since M/F is a Galois extension, Lemma 1.9 shows that $\mathrm{Gal}(M/F)$ contains an element of order 8. Thus $M \nsubseteq F^{(3)}$ by Lemma 4.1, but $M = K(\sqrt[4]{\sqrt{a}},\sqrt{-1},\sqrt{2}) \subseteq K^{(3)}$.

Now suppose that $K = F(\sqrt{-1})$, that $[K : F] = 2$, and that F is not Pythagorean. Then there exists $\beta \in K$ such that $\beta\bar{\beta} = a \notin \dot{F}^2$, where $\bar{\beta} = \sigma(\beta)$ and $\langle\sigma\rangle = \mathrm{Gal}(K/F)$. Let $f(x) = (x^4 - \beta)(x^4 - \bar{\beta}) \in F[x]$. Then $M = F(\sqrt{-1}, \sqrt[4]{\beta}, \sqrt[4]{\bar{\beta}})$ is a splitting field of f over F. Observe that $\sqrt{a} = \sqrt{\beta}\sqrt{\bar{\beta}} \in M$ and $\sqrt{-a} = \sqrt{-1}\sqrt{a} \in M$. Consider the following tower of extensions:

$$F \subseteq F' = F(\sqrt{-a}) \subseteq K' = F(\sqrt{-1},\sqrt{a}) \subseteq K'(\sqrt{\beta}) \subseteq K'(\sqrt[4]{\beta}) \subseteq M.$$

We have $[K' : F'] = 2$, since $\sqrt{-1} \in F(\sqrt{-a})$ would imply that $-1 \in \dot{F}^2 \cup -a\dot{F}^2$, but $-1 \notin \dot{F}^2$ and $a \notin \dot{F}^2$. Also, $[K'(\sqrt{\beta}) : K'] = 2$, since $\sqrt{\beta} \in K' = K(\sqrt{a})$ would imply that $\beta \in \dot{K}^2 \cup a\dot{K}^2$, which would in turn imply that $a = \beta\bar{\beta} \in \dot{F}^2 \cup a^2\dot{F}^2 = \dot{F}^2$, but $a \notin \dot{F}^2$. Since $x^2 - \beta$ is irreducible over K', it follows that $x^4 - \beta$ is irreducible over K' (again by [**5**, Theorem 9.1, p. 297]). Therefore $[K'(\sqrt[4]{\beta}) : K'] = 4$ and $K'(\sqrt[4]{\beta})/K'$ is Galois with $\mathrm{Gal}(K'(\sqrt[4]{\beta})/K') \cong \mathbb{Z}/4\mathbb{Z}$. Since $[K'(\sqrt{\beta}) : F'] = 4$, $K' = F'(\sqrt{a}) = F'(\sqrt{-1})$, and $N_{K'/F'}(\beta) = a$, it follows that $K'(\sqrt{\beta})/F'$ is Galois with $\mathrm{Gal}(K'(\sqrt{\beta})/F') \cong \mathbb{Z}/4\mathbb{Z}$. Since M/F is a Galois extension, Lemma 1.9 again implies that $\mathrm{Gal}(M/F)$ contains an element of order 8. This implies that $M \nsubseteq F^{(3)}$ by Lemma 4.1, but $M \subseteq K^{(3)}$ since $K(\sqrt[4]{\beta})/K$ and $K(\sqrt[4]{\bar{\beta}})/K$ are both cyclic quartic extensions. □

We finish with two propositions that sharpen Lemma 4.1. Recall that a field F is *Euclidean* if F is formally real and $|\dot{F}/\dot{F}^2| = 2$.

Proposition 4.4. *The following statements are equivalent:*

(1) $\mathrm{Gal}(F^{(3)}/F)$ *has exponent* 2*;*
(2) *F admits no Galois extension with Galois group isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or D_4, the dihedral group of order* 8*, but F is not quadratically closed;*
(3) *F is Pythagorean and $|\dot{F}/\dot{F}^2| = 2$;*
(4) *F is Euclidean;*
(5) *$-1 \notin F^2$ and $F^{(2)} = F(\sqrt{-1})$ is quadratically closed;*
(6) $\mathrm{Gal}(F^{(3)}/F) \cong \mathbb{Z}/2\mathbb{Z}$*;*
(7) *$F^{(2)} = F^{(3)}$ and F is not quadratically closed.*

Proof. (3) ⇒ (4). If F were nonreal, then every element of F would be a sum of squares, and hence a square. That would imply that $|\dot{F}/\dot{F}^2| = 1$, a contradiction.

(4) ⇒ (3). We must have $F = F^2 \cup -F^2$. Thus $a^2 + b^2 \in F^2$ for all $a, b \in F$ since $a^2 + b^2$ cannot be negative. Therefore F is Pythagorean.

(3) ⇒ (5). Since F must be formally real, we have $-1 \notin F^2$, and thus $F^{(2)} = F(\sqrt{-1})$. Apply Corollary 1.4 with $K = F(\sqrt{-1})$ to conclude that $F(\sqrt{-1})$ is quadratically closed.

(5) ⇒ (6). Since $F^{(2)} = F(\sqrt{-1})$ is quadratically closed, we have $F^{(3)} = F^{(2)}$, and thus $[F^{(3)} : F] = 2$.

(6) ⇒ (7). We have $F \subseteq F^{(2)} \subseteq F^{(3)}$ and $[F^{(3)} : F] = 2$. Since F cannot be quadratically closed, we must have $F^{(2)} = F^{(3)}$.

(7) ⇒ (1). $\mathrm{Gal}(F^{(3)}/F) = \mathrm{Gal}(F^{(2)}/F)$, and the latter group has exponent 2.

(1) ⇒ (2). If $\mathrm{Gal}(L/F)$ is isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or D_4, then $L \subseteq F^{(3)}$. But $\mathrm{Gal}(L/F)$ is a homomorphic image of $\mathrm{Gal}(F^{(3)}/F)$, and hence cannot have exponent 4.

(2) ⇒ (3). Let K be any quadratic extension of F, and suppose that L is a quadratic extension of K. Let E/F be a Galois closure of L/F. The hypothesis on F implies that $E = L$ and $\mathrm{Gal}(L/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since every quadratic extension of K comes from F, it follows that the map $\dot{F}/\dot{F}^2 \to \dot{K}/\dot{K}^2$ is surjective. Corollary 1.4 implies that F is Pythagorean and $K = F(\sqrt{-1})$. Since K is uniquely determined, it follows that $|\dot{F}/\dot{F}^2| = 2$. □

Proposition 4.5. *The following statements are equivalent:*

(1) $\mathrm{Gal}(F^{(3)}/F)$ *has exponent* 1*; that is,* $F^{(3)} = F$*;*

(2) $F^{(2)} = F$;
(3) *F is quadratically closed.*

Proof. (1) $\Rightarrow$ (2). This follows from $F \subseteq F^{(2)} \subseteq F^{(3)}$.
It is obvious that statements (2) and (3) are equivalent.
(2) $\Rightarrow$ (1). This follows from $F \subseteq F^{(3)} \subseteq F^{\{3\}} = (F^{(2)})^{(2)} = F^{(2)} = F$. $\square$

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Dept. of Mathematics
University of Kentucky
Lexington, Kentucky 40506-0027
USA

leep@ms.uky.edu

Dept. of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221-0025
USA

tsmith@math.uc.edu