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Mathematical Notes

A remark on the values of the Riemann zeta function

Ján Mináč

Abstract. We are connecting the values of the Riemann zeta function $\zeta(s)$ at all nonpositive integers a with the partial sums $S_a(M) = \sum_{n=1}^{M-1} n^a$ as $\int_0^1 S_a(x)\,dx = \zeta(-a)$. We shall prove this relationship in two ways: one of them uses Bernoulli numbers, and the other uses the formula in "Landau's Handbuch", which relates to the values of the ζ function.

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The Riemann zeta function $\zeta(s), s = \sigma + it$, is the analytical continuation of the function $f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \sigma > 1$, to the whole plane except where $s = 1$ where $\zeta(s)$ has a simple pole. (See e.g. [Tit] or [I-R], Chapter 16.)

In particular, one may consider $\zeta(-a)$ for each $a \in \mathbb{N}$. The remarkable fact is that all of the values of $\zeta(-1) = -1/12$, $\zeta(-2) = 0$, $\zeta(-3) = 1/120$, $\zeta(-4) = 0$, $\zeta(-5) = -1/252$, etc., are rational numbers.

The question becomes: Is there any connection between the values of $\zeta(-a)$ and the partial sums $S_a(M) := \sum_{n=1}^{M-1} n^a$?

It is well-known that $S_a(M)$ are expressible as polynomials in M, of the degree $a+1$ with rational coefficients. For example $S_1(M) = \frac{M(M-1)}{2}$. It seems that the following interesting fact hasn't been recorded yet.

Fact.

$$\zeta(-a) = \int_0^1 S_a(x)\,dx, \quad \textit{for each} \quad a \in \mathbb{N}.$$

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Examples.

$$\zeta(-1) = \int_0^1 \frac{x(x-1)}{2}\,dx = -\frac{1}{12}$$

$$\zeta(-2) = \int_0^1 \frac{x(x-1)(2x-1)}{6}\,dx = 0$$

$$\zeta(-3) = \int_0^1 \frac{(x(x-1))^2}{4}\,dx = \frac{1}{120}$$

$$\zeta(-4) = \int_0^1 \frac{1}{5}\left(x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x\right)dx = 0$$

$$\zeta(-5) = \int_0^1 \frac{x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{x^2}{2}}{6}\,dx = -\frac{1}{252}, \text{ etc.}$$

The formula above holds true also for $a = 0$. Indeed, then it is well-known that $\zeta(0) = -1/2 = \int_0^1 (x-1)dx$.

In the proof of the Fact above, we have used the following well-known facts about Bernoulli numbers and polynomials. (See [I-R], Chapter 15, including Exercises 12–17 on Page 248.)

Let B_k be the k^{th} Bernoulli number for each $k = 0, 1, 2, \dots$ and $B_0(x) = 1$, $B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}$ the m^{th} Bernoulli polynomial for each $m \in \mathbb{N}$. Then $B_1 = -1/2$, $0 = B_3 = B_5 = \cdots$, $B_m(1) = (-1)^m B_m = (-1)^m B_m(0)$, $\frac{d}{dx}B_m(x) = mB_{m-1}(x)$ for each $m \in \mathbb{N}$. Finally $S_a(M) = \frac{B_{a+1}(M) - B_{a+1}(1)}{a+1}$, and $\zeta(-a) = (-1)^a \frac{B_{a+1}}{a+1} = -\frac{B_{a+1}}{a+1}$ for each $a \in \mathbb{N}$. (See [I-R], Chapter 16.)

The proof of the fact. Let $a \in \mathbb{N}$. Then

$$\int_0^1 S_a(x)\,dx = \int_0^1 \frac{B_{a+1}(x) - B_{a+1}(1)}{a+1}\,dx$$
$$= (-1)^a \frac{B_{a+1}}{a+1} + \frac{B_{a+2}(1) - B_{a+2}(0)}{(a+1)(a+2)}$$
$$= (-1)^a \frac{B_{a+1}}{a+1} = \zeta(-a).$$

Indeed if a is even then $B_{a+2}(1) - B_{a+2}(0) = B_{a+2}(1) - B_{a+2} = 0$ and if a is odd, then both $B_{a+2}(1)$ and $B_{a+2}(0)$ are 0. ■

There are, of course, many other ways in which to see this formula. One way is to use the beautiful formula

$$(s-1)(\zeta(s)-1) - 1 = -\sum_{q=1}^{\infty} \frac{(s-1)s\cdots(s+q-1)}{(q+1)!}(\zeta(s+q)-1) \tag{1}$$

([Lan], page 274).

Using this formula, and the fact that $\zeta(s)$ has the unique simple pole $s = 1$, $\lim_{s\to 1}(s-1)\zeta(s) = 1$, one can prove our statement without using Bernoulli numbers. Indeed for each $a = 1, 2, \dots$ we obtain from (1)

$$\frac{(-1)^{a+1}}{a+2} = \sum_{k=0}^{a} \binom{a+1}{k+1} (-1)^k \zeta(k-a). \tag{2}$$

On the other hand we have for $a = 0, 1, \dots$ and $M = 2, 3, \dots$

$$\begin{aligned}(M-1)^{a+1} &= 1 + (2^{a+1} - 1) + (3^{a+1} - 2^{a+1}) + \cdots \\ &\quad + ((M-1)^{a+1} - (M-2)^{a+1}) \\ &= 1 + \sum_{n=2}^{M-1} (n^{a+1} - (n-1)^{a+1}) \\ &= 1 + \sum_{n=2}^{M-1} \left((a+1)n^a - \binom{a+1}{2} n^{a-1} + \cdots + (-1)^a\right) \\ &= \sum_{k=0}^{a} \binom{a+1}{k+1} (-1)^k S_{a-k}(M).\end{aligned}$$

Here we use the identity

$$0 = 1 - (a+1) + \binom{a+1}{2} - \binom{a+1}{3} - \cdots + (-1)^{a+1}.$$

Plugging in x instead of M, and taking the integral from 0 to 1, we obtain,

$$\frac{(-1)^{a+1}}{a+2} = \sum_{k=0}^{a} \binom{a+1}{k+1} (-1)^k \int_0^1 S_{a-k}(x)\,dx. \tag{3}$$

Using (2), (3) and the fact that $\int_0^1 S_0(x)\,dx = \zeta(0)$ we conclude: $\zeta(-a) = \int_0^1 S_a(x)\,dx$ for all $a \in \mathbb{N} \cup \{0\}$.

It would be interesting to develop some similar formulas for some other values of the Riemann zeta function. It would also be exciting to extend this formula for other zeta functions.

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The University of Western Ontario
London, Ontario Canada
N6A 5B7